1. Here we will apply a simplified version of Multiple-Timescale Analysis to the problem of particle motion in constant, uniform \( E \) and \( B \) fields.

As done in Lecture #3, we assume a right-handed, orthonormal basis aligned with the direction of the magnetic field \((\hat{e}_1, \hat{e}_2, \hat{b})\) such that \( \hat{e}_1 \times \hat{e}_2 = \hat{b} \). The Lorentz Force Law is

\[
m\frac{dv}{dt} = q(E + v \times B)
\]

for an electric field \( E = E_1\hat{e}_1 + E_2\hat{e}_2 + E_\parallel\hat{b} \) and a magnetic field \( B = B_0\hat{b} \). For this problem, we will take the case \( E_\parallel = 0 \).

(a) First, let us convert the dimensional form of the Lorentz Force Law above to a dimensionless equation. Derive the dimensionless form

\[
\frac{dv'}{dt'} = E' + v' \times \hat{b}
\]

for dimensionless quantities \( t' = \omega_0 t \), \( v' = v/v_\perp \), and \( E' = \frac{E}{B_0v_\perp} \) where \( v_\perp = \sqrt{v_1^2 + v_2^2} \).

(b) Verify that the quantity \( E' = |E'| \) is dimensionless (in the SI system of units).

(c) Show that the condition \( E' \ll 1 \) means that the \( E \times B \) drift is slow compared to the perpendicular velocity, \( |v_E| \ll v_\perp \).

(d) Assuming \( E' \ll 1 \), the timescales of the Larmor motion and the \( E \times B \) drift are well separated. For the expansion parameter, take \( \epsilon = E' \ll 1 \). As an aid in the bookkeeping for the order of magnitude of each term, we can add an \( \epsilon \) to the electric field term in our equation to remind us of its order,

\[
\frac{dv'}{dt'} = \epsilon E' + v' \times \hat{b}
\]

We’ll assume a fast timescale \( t' \) and a slow timescale \( \tau' = \epsilon t' \). Decompose the total velocity into rapidly varying piece \( v'_1 \) and a smaller slowly varying piece \( v'_2 \), \( v' = v'_1(t') + \epsilon v'_2(\tau') \).

Write down the expansion of \( d/dt' \) assuming two timescales.

(e) Derive the equation at \( O(1) \) and solve for \( v'_1(t') \) given the (dimensional) initial conditions at \( t = 0 \) of \( v = v_\perp\hat{e}_1 + v_\parallel\hat{b} \).

(f) Derive the equation at \( O(\epsilon) \). Solve for \( v'_2(\tau') \). HINT: Do not forget to treat \( t' \) and \( \tau' \) as independent variables.

(g) Sum the solution for each order to get the total solution \( v'(t', \tau') \). Convert back to dimensional form to yield the final, complete solution \( v(t) \).

2. A cylindrical column of plasma rotates around its central axis (as though it were a rigid solid) at an angular velocity \( \omega_0 \). A constant uniform magnetic field \( B_0 \) is present parallel to the axis of rotation.

(a) Assuming that the rigid rotation can be described by \( v_E = (\omega_0\hat{z}) \times r \), where \( v_E = E \times B/B^2 \), compute the electric field \( E \) in the plasma column. Use cylindrical \((r, \phi, z)\) coordinates.

(b) Is there a polarization charge \( \rho = \epsilon_0 \nabla \cdot E \) associated with this electric field? If so, how does \( \rho \) depend on the distance from the central axis?

(c) Find the electrostatic potential \( \phi \) such that \( E = -\nabla \phi \).

(d) How would you induce a motion of this type in a magnetized plasma column?