29:194 Homework #7

Reading: Chen, Sec 3.4-3.6 (p.68–77), Sec 5.7 (p 184–186), Sec 4.1-4.3 (p.79–86)

Due at the beginning of class, Thursday, October 30, 2008.

1. Beginning with the Adiabatic Equation of State, derive the following alternative equation for the evolution of the pressure

$$\frac{dp}{dt} = -\gamma p \nabla \cdot \mathbf{U}$$

2. Magnetic Pressure vs. Magnetic Tension

   (a) For a magnetic field of the form $\mathbf{B} = B_0 y/L\hat{\mathbf{z}}$, calculate the force density due to the magnetic pressure term and magnetic tension term in the simplified form of the Momentum Equation.

   (b) Calculate the magnetic tension and magnetic pressure force densities for a wave-like magnetic field of form $\mathbf{B} = B_0\hat{\mathbf{z}} + B_1 \sin(k_\parallel z)\hat{\mathbf{y}}$. You may take $B_1 \ll B_0$ and drop terms that are second-order in B_1.

3. A dipole magnetic field has the form

$$\mathbf{B} = \frac{\mu_0 M}{4\pi}\frac{1}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

   We can represent $\mathbf{B}$ in Clebsch form $\mathbf{B} = \nabla\psi \times \nabla\phi$, where ϕ is the azimuthal coordinate and ψ is a scalar function $\psi(r, \theta)$.

   (a) Find $\psi(r, \theta)$.

   (b) Use this result to easily calculate the equation for a magnetic field line in the form $r = r(\theta)$.
   HINTS: Choose an appropriate way to label the field line *and* look back at problem 3 on HW#3.

4. Consider the Clebsch representation of a magnetic field $\mathbf{B} = \nabla\alpha \times \nabla\beta$. Show that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B})$$

   is satisfied if the Clebsch variables satisify

$$\frac{\partial \alpha}{\partial t} + \mathbf{U} \cdot \nabla\alpha = 0 \quad \text{and} \quad \frac{\partial \beta}{\partial t} + \mathbf{U} \cdot \nabla\beta = 0.$$

5. Star Formation
   Suppose the interstellar medium has a number density of 10^6 m^{-3} and a straight, uniform magnetic field of magnitude $B = 3 \times 10^{-10}$ T.

   (a) Calculate the field strength in a star if the flux remains “frozen-in” and the star forms by a spherical collapse to the radius and mass of the sun.

   (b) Compare the force densities at the surface of the star due to the magnetic pressure and to gravity, where the Momentum Equation with the gravitational force added is

$$\rho\frac{\partial \mathbf{U}}{\partial t} + \rho\mathbf{U} \cdot \nabla\mathbf{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{\mu_0} - \rho\nabla\Phi_G$$

   where the gravitational potential energy is $\Phi_G = -GM/r$ outside of a star of mass M.