Lecture #11 Collisions and Resistivity

I. Single Particle Motion and Collisions

A. 1. So far, we have considered the motion of a single charged particle in prescribed (non-self-consistent) $\underline{E}$ & $\underline{B}$ fields

2. Another effect that can affect the motion of a particle is the collision with another particle.

a. This is not a collective effect, such as the collective motion of ions & electrons producing current and charge densities and leading to $\underline{E}$ & $\underline{B}$ fields

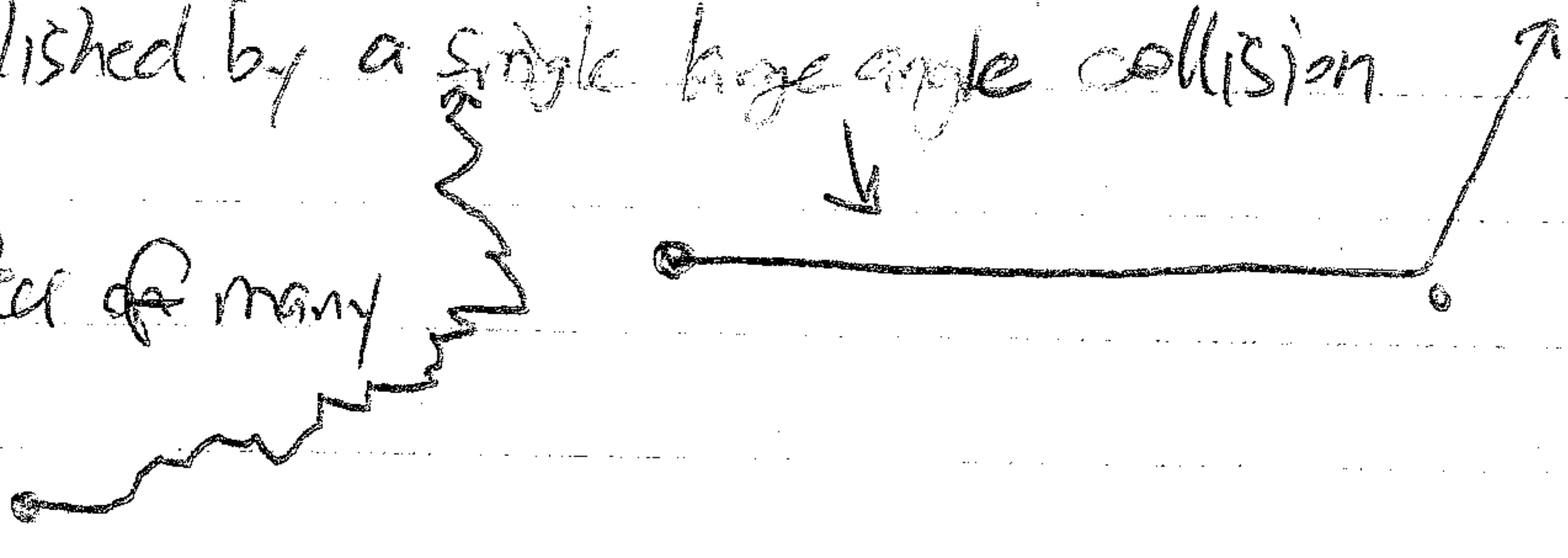
b. Although a single charged particle may collide with many other particles (as we shall see) these interactions are independent, and do not act cooperatively, so collisions belongs with single particle motion discussion

II. Single Large Angle vs. Many Small Angle Collisions.

A. Def: Collision time $\tau \equiv$ Time required for particle trajectory to be deflected by $\frac{\pi}{2}$.

1. This may be accomplished by a single large angle collision

2. Or by the summed effect of many small angle collisions.



3. We will see, in fact, the small angle collisions dominate.

$\Rightarrow$ Coulomb force is long range, so particle can interact with many particles at once

$\Rightarrow$ But Debye shielding limits long-range interactions, leaving possible interactions with N_D particles within Debye sphere.

1. ⑥ (Continued)

$$= \frac{z_1^2}{L_1}\sqrt{B_1}\int_{-L_1}^{L_1}\sqrt{1-\left(\frac{z}{z_1}\right)^2}\,\frac{dz}{z_1} = \frac{z_1^2}{L_1}\sqrt{B_1}\int_{-1}^{1}\sqrt{1-w^2}\,dw$$

$w=\frac{z}{z_1}$ $\quad dw=\frac{dz}{z_1}$

⑦ Substituting $z_1^2 = L_1^2\left(\frac{B_{t_1}}{B_1}-1\right)$, we get:

$$L_0\sqrt{B_0}\int_{-1}^{1}\sqrt{1-y^2}\,dy = L_1\sqrt{B_1}\left(\frac{B_{t_1}}{B_1}-1\right)\int_{-1}^{1}\sqrt{1-w^2}\,dw$$

⑧ The integrals cancel, leaving the relation implied by the second adiabatic invariant

$$L_0\sqrt{B_0} = L_1\sqrt{B_1}\left(\frac{B_{t_1}}{B_1}-1\right)$$

a. In this case $L_1 = L_0$ and $B_1 = 2B_0$, so

$$\cancel{L_0}\sqrt{\cancel{B_0}} = \cancel{L_0}\sqrt{2\cancel{B_0}}\left(\frac{B_{t_1}}{2B_0}-1\right) \Rightarrow \frac{1}{\sqrt{2}} = \frac{B_{t_1}}{2B_0}-1$$

$$B_{t_1} = 2B_0\left(1+\frac{1}{\sqrt{2}}\right) = \boxed{B_0(2+\sqrt{2}) = B_{t_1}}$$

$$z_1 = \pm L_0\sqrt{\frac{\cancel{B_0}(2+\sqrt{2})}{2\cancel{B_0}}-1} = \pm L_0\sqrt{\cancel{1}+\frac{1}{\sqrt{2}}\cancel{-1}} = \boxed{\frac{\pm L_0}{2^{\frac{1}{4}}} = z_1}$$

b. Take $L_1 = \frac{L_0}{2}$ $\quad B_1 = 2B_0$

$$\cancel{L_0}\sqrt{\cancel{B_0}} = \frac{\cancel{L_0}}{2}\sqrt{2\cancel{B_0}}\left(\frac{B_{t_1}}{2B_0}-1\right) \Rightarrow \sqrt{2} = \frac{B_{t_1}}{2B_0}-1$$

$$\boxed{B_{t_1} = 2B_0(1+\sqrt{2})}$$

$$z = \pm\left(\frac{L_0}{2}\right)\sqrt{\frac{\cancel{2B_0}(1+\sqrt{2})}{\cancel{2B_0}}-1} = \pm\frac{L_0}{2}\sqrt{\cancel{1}+\sqrt{2}\cancel{-1}} = \frac{\pm 2^{\frac{1}{4}}L_0}{2} = \boxed{\pm\frac{L_0}{2^{3/4}} = z_1}$$

Hawes 27b

Homework #4 solutions:

2. (a)

32	0.83855
64	0.36003
128	0.16661
256	0.080143
512	0.039304
1024	0.019463
2048	0.0096849
4096	0.0048308
8192	0.0024125
16384	0.0012055

(b)

32	0.041902
64	0.010186
128	0.0025291
256	0.00063119
512	0.00015773
1024	3.9428e-05
2048	9.8567e-06
4096	2.4642e-06
8192	6.1604e-07
16384	1.5401e-07

(c) See attached plot

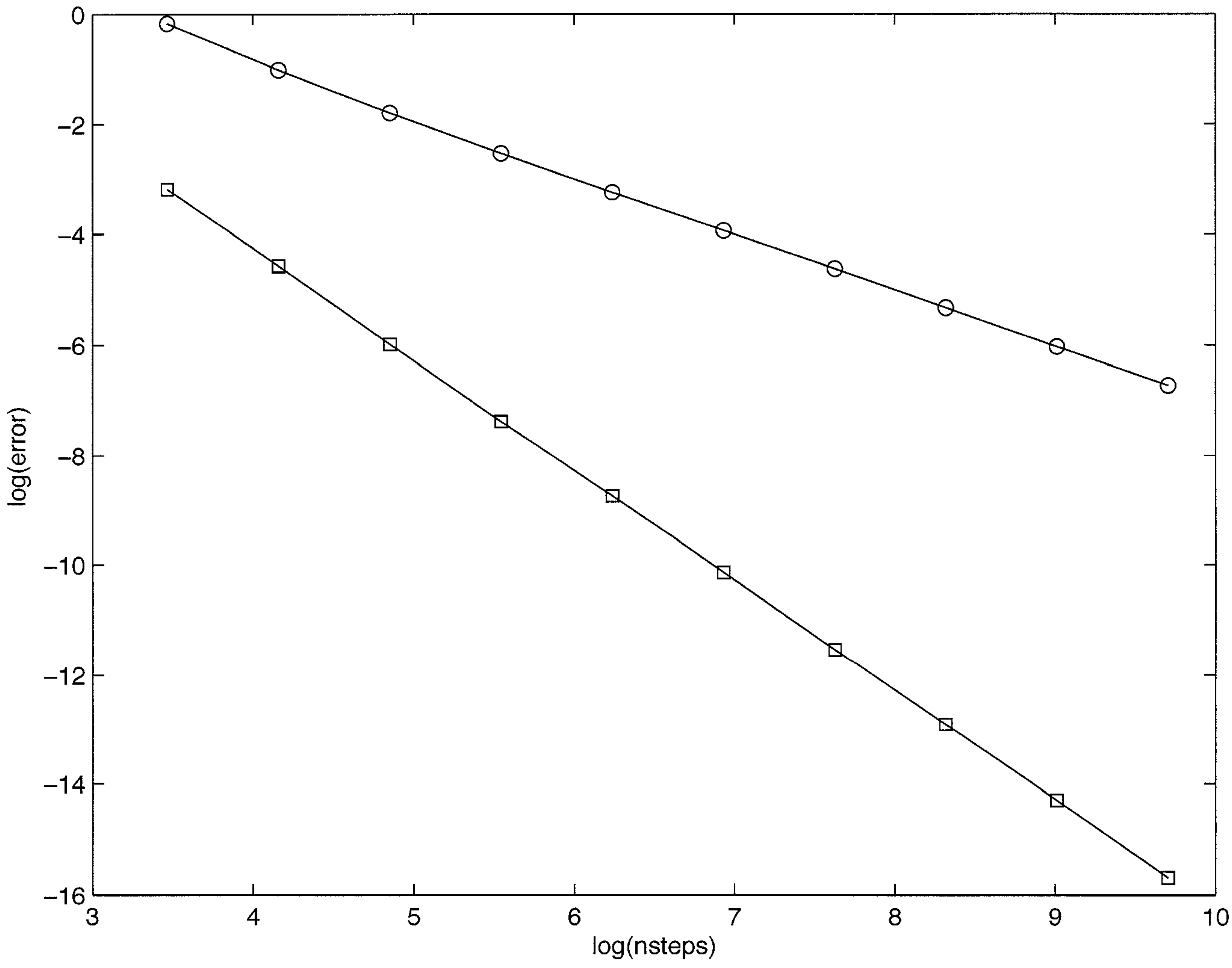
(d) The slope of the euler method is -1
The slope of the leapfrog method is -2

(e) The RK45 method requires 101 steps, giving an error 3.6019E-06. This mehtod is better than Euler's method by more than a factor of 160 (since the error from Euler's method at 16384 steps is still greater than the error from RK45). The leapfrog method achieves a comparable error at 4096 steps, so RK45 is about 40 times better in terms of the number of timesteps required.

HW#4 2.c.

Haines (2-c)

Error vs. Number of Timesteps



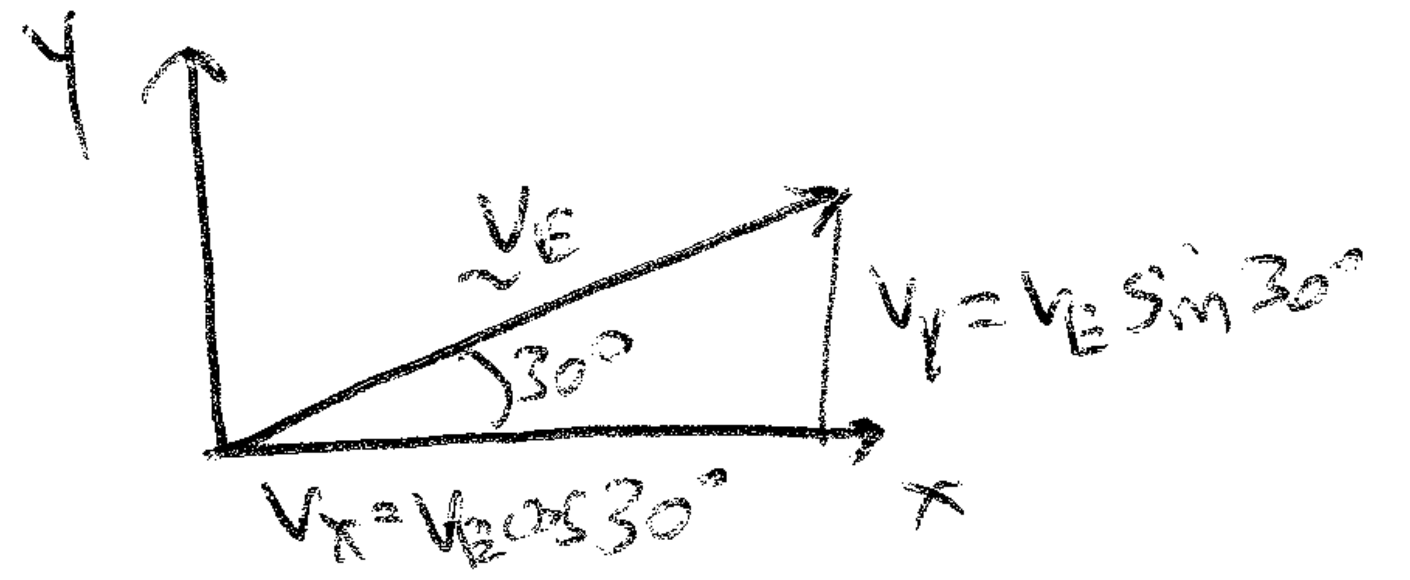
HW#4 (Continued) Haves (3)

NOTE: 3(a) should have asked for a drift of r_L in time of 10 Larmor orbits

3.a. We want a drift $|V_E| = \frac{r_L}{10T}$

where $T = \frac{2\pi}{\omega_c}$ $\Rightarrow |V_E| = \frac{r_L \omega_c}{10 \cdot 2\pi} = \frac{v_\perp}{20\pi}$

We want $\underset{\sim}{V_E} = \frac{1}{B}(\underset{\sim}{E} \times \hat{\underset{\sim}{b}}) = \frac{1}{B}(E_y \hat{x} - E_x \hat{y}) = V_E \cos 30° \hat{x} + V_E \sin 30° \hat{y}$



Thus $E_x = -B v_E \sin 30°$

$E_y = B v_E \cos 30°$

$$\boxed{\begin{aligned} E_x &= -(1)\left(\frac{(1)}{20\pi}\right)\sin 30° = -0.00796 \\ E_y &= (1)\left(\frac{(1)}{20\pi}\right)\cos 30° = 0.0138 \end{aligned}}$$

(b) See attached Plot

4.a. $B = (0, 0, 1 - \alpha r)$, so $|B| = 1 - \alpha r$

$$\underset{\sim}{V}_{\nabla B} = \frac{-v_\perp^2}{2\omega_c} \frac{\nabla B \times \underset{\sim}{B}}{B_0^2}$$

In cylindrical coordinates,

$$\nabla B = \frac{\partial B}{\partial r}\hat{r} + \cancel{\frac{1}{r}\frac{\partial B}{\partial \phi}\hat{\phi}} + \cancel{\frac{\partial B}{\partial z}\hat{z}} = \frac{\partial}{\partial r}(1 - \alpha r)\hat{\underset{\sim}{r}} = -\alpha \hat{\underset{\sim}{r}}$$

Thus $\underset{\sim}{V}_{\nabla B} = -\frac{v_\perp^2}{2\omega_c} \frac{\nabla B \times \underset{\sim}{B}}{B_z^2} = \frac{-v_\perp^2}{2\omega_c} \frac{(-\alpha \hat{r}) \times B \hat{\underset{\sim}{z}}}{B^2} = \frac{v_\perp^2 \alpha}{2\omega_c B}(-\hat{\phi})$

$$\boxed{-\hat{\phi} \text{ direction for } \underset{\sim}{V}_{\nabla B}}$$

b. We'll solve this problem approximately by estimating the position of the Guiding Center.

II. (Continued)

E. Collisional Equilibration Times:

1. Collision frequency for species s on species r

$$\nu_{sr} = \frac{e^4 n_{0r}}{2^{5/2}\pi\epsilon_0^2 m_s^{1/2} (kT_s)^{3/2}} \ln N_D$$

2. Electron-Ion collisions: ν_{ei} calculated as before.

3. Electron-electron collisions:

a. Need to transform to center-of-mass frame. May introduce a few factors of 2, but otherwise $\boxed{\nu_{ee} \simeq \nu_{ei}}$

4. Ion-Ion collisions:

a. Same as electron-electron collisions, except we must replace m_e by m_i in denominator (taking $T_i = T_e$)

$$\boxed{\nu_{ii} = \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \nu_{ee}}$$

5. Ion-electron collisions:

a. Center-of-mass frame calculation introduces another factor of $\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}$, so $\boxed{\nu_{ie} \simeq \left(\frac{m_e}{m_i}\right) \nu_{ee}}$

NOTE: For proton-electron plasma $\boxed{m_i/m_e = 1836.}$

6. For a plasma with arbitrary velocity distributions for both protons & electrons and unequal temperatures $T_i \neq T_e$,

a. Electrons thermalize on timescale $\tau_{ee} \sim \frac{1}{\nu_{ee}} \sim \frac{1}{\nu_{ei}}$

b. Ions thermalize on timescale $\tau_{ii} \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \tau_{ee} \simeq 43\,\tau_{ee}$

c. Ions & electrons come to same temperature $\tau_{ie} \sim \frac{m_i}{m_e} \tau_{ee} = 1836\,\tau_{ee}$

III. Resistivity and Collisions:

A. Consider an unmagnetized, quasineutral plasma of ions and electrons.

1. In response to an applied Electric field $\underset{\sim}{E}$, ~~the~~ a current will flow in the plasma.

   a. Current density $\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{v}_s = n_{0i} e \underset{\sim}{v}_i - n_{0e} e \underset{\sim}{v}_e$ $\quad \boxed{\underset{\sim}{j} = e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e)}$

   b. For equivalent temperatures (or energies)

   $$\frac{1}{2} m_e v_e^2 = \frac{1}{2} m_i v_i^2 \Rightarrow v_e = \left(\frac{m_i}{m_e}\right)^{1/2} v_i$$

   For protons and electrons $v_e \simeq 43\, v_i$

   c. Thus, $\boxed{\text{Current in a plasma is carried mostly by electrons.}}$

2. a. Because of conservation of momentum, electron-electron collisions do not lead to resistivity.

   b. ~~The~~ Electron-ion collisions are responsible for resistivity.

3. Electron Momentum Equation (in Unmagnetized plasma) $n_0 = n_e = n_i$

   a. $$m_e n_0 \frac{dv_e}{dt} = -e n_0 \underset{\sim}{E} + \underbrace{m_e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e) \nu_{ei}}_{\text{Collisional term}}$$

   b. In steady state, $\frac{dv_e}{dt} = 0$, so

   $$\underset{\sim}{E} = \frac{m_e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e) \nu_{ei}}{+e n_0} = \frac{\overbrace{e n_0 (\underset{\sim}{v}_i - \underset{\sim}{v}_e)}^{=\underset{\sim}{j}}\, m_e \nu_{ei}}{e^2 n_0} = \left(\frac{m_e \nu_{ei}}{e^2 n_0}\right) \underset{\sim}{j}$$

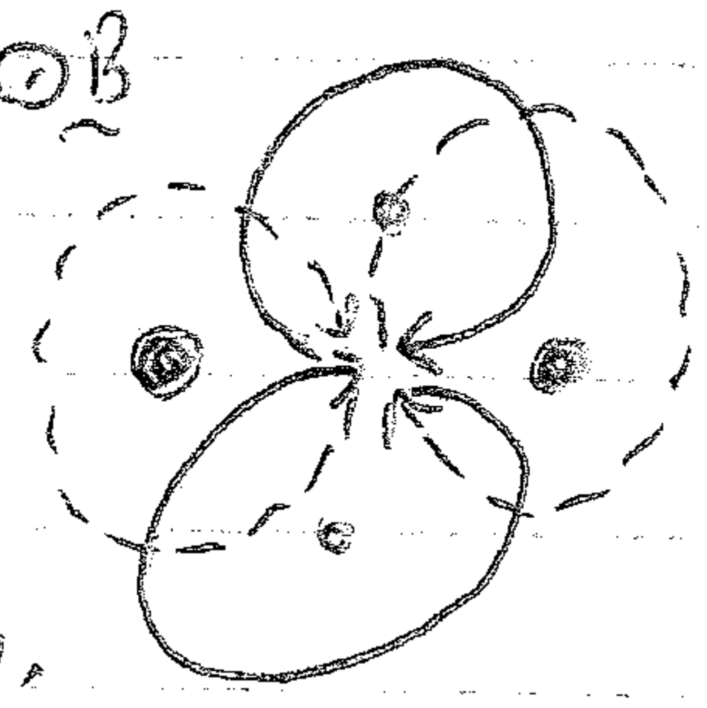
   c. $\boxed{\text{Ohm's Law} \quad \underset{\sim}{E} = \eta \underset{\sim}{j}}$ $\quad$ where $\eta = \dfrac{m_e \nu_{ei}}{e^2 n_0}$ is the Resistivity (specific)

III A. (Continued)

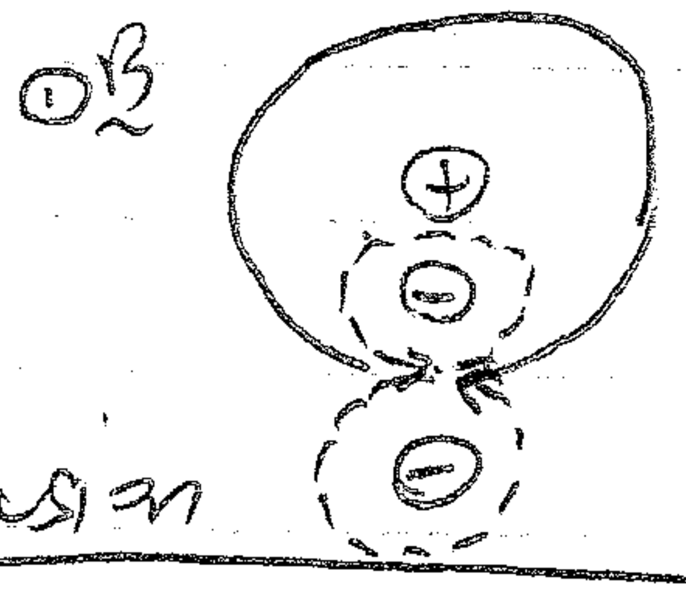
4. $$\eta = \frac{m_e}{e^2 \cancel{N_0}}\left[\frac{\cancel{N_0}\, e^4 \cdot \ln N_0}{2^{5/2}\pi\epsilon_0^2 m_e^{1/2}(kT_e)^{3/2}}\right] = \boxed{\frac{e^2 m_e^{1/2} \ln N_0}{2^{5/2}\pi\epsilon_0^2 (kT_e)^{3/2}} = \eta}$$

a. Resistivity is independent of density!
b. Resistivity decreases with increasing temperature!

IV. Collisions and Magnetic Confinement



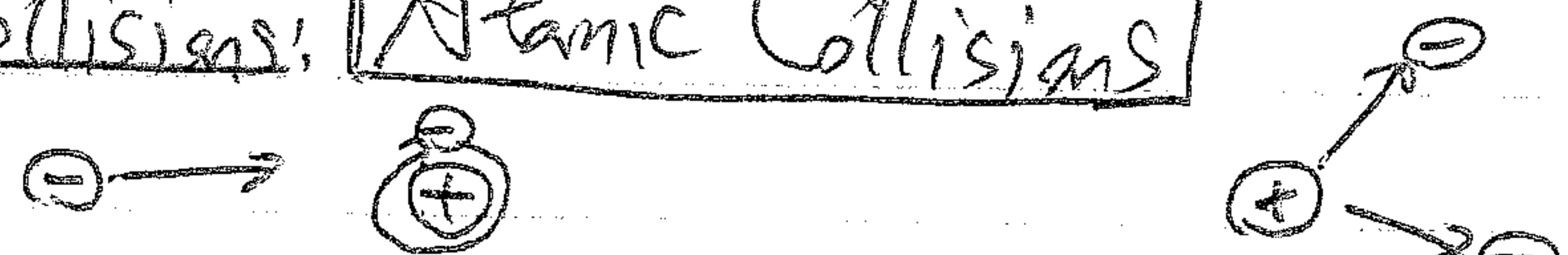
A. Like-Particle Collisions: Center-of-mass remains stationary
⇒ Like-particle collisions give little diffusion across magnetic field lines.



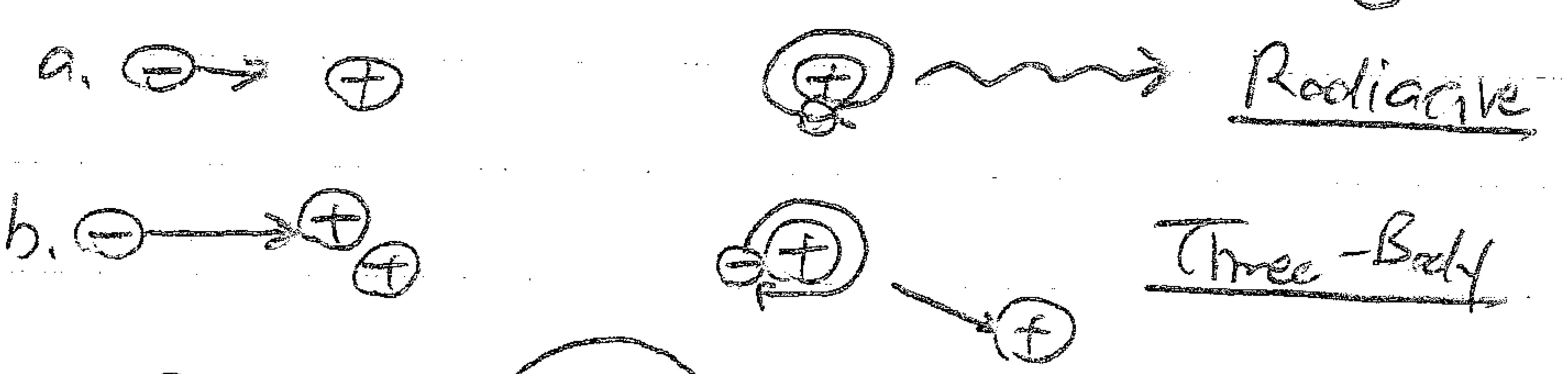
B. Unlike-Particle Collisions:
Center-of-mass is shifted
⇒ Unlike particle collisions give rise to diffusion across magnetic field lines ⇒ $\boxed{\text{LOSS OF CONFINEMENT}}$

V. Other Types of Collisions: $\boxed{\text{Atomic Collisions}}$

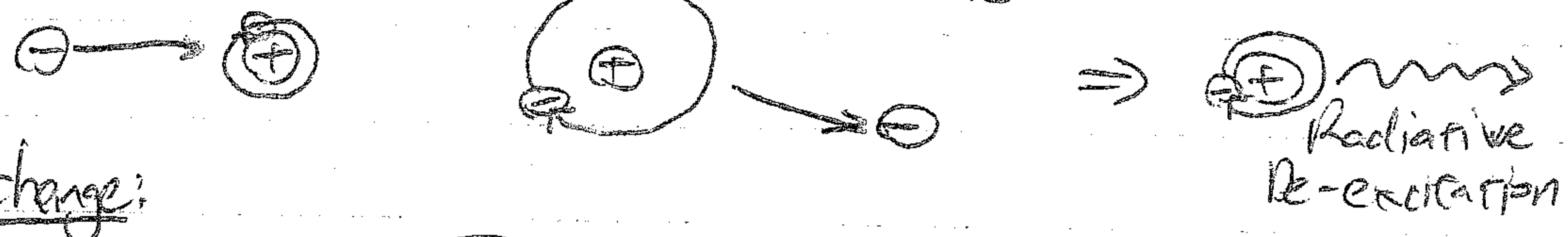
1. Ionization:



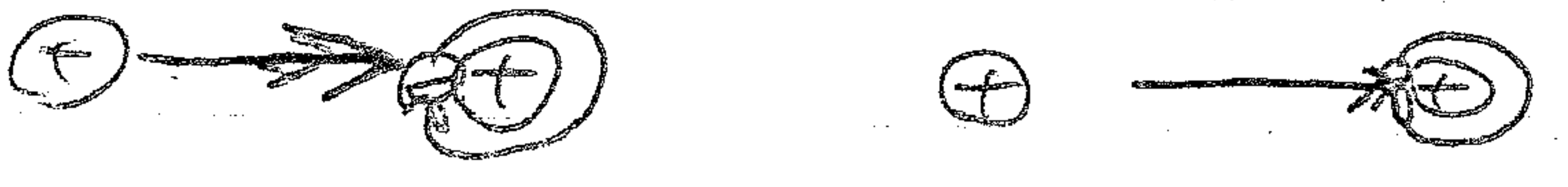
2. Recombination:



3. Excitation:



4. Charge Exchange:



5. Photoionization:

6. Elastic ~~[illegible]~~