Lecture #12 Kinetic Description of a Plasma

I. Overview of Plasma Descriptions

Lorentz Force Law

$$m\frac{d\underline{v}}{dt} = q(\underline{E} + \underline{v}\times\underline{B})$$

$\underline{E}$ & $\underline{B}$ $\Rightarrow$ $\underline{x}$ & $\underline{v}$

Particles: Ions and Electrons

$\underline{x}$, $\underline{v}$ $\Rightarrow$ ρ, $\underline{j}$

$\underline{B}$ & $\underline{E}$ fields

Maxwell's Equations

$$\nabla\cdot\underline{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla\times\underline{B} = \mu_0\underline{j} + \epsilon_0\mu_0\frac{\partial\underline{E}}{\partial t}$$

$$\nabla\times\underline{E} = -\frac{\partial\underline{B}}{\partial t}$$

$$\nabla\cdot\underline{B} = 0$$

ρ & $\underline{j}$ $\Rightarrow$ $\underline{E}$ & $\underline{B}$

A. Inconsistent Models:

1. Single Particle Motion: Lorentz Force Law for specified $\underline{E}$ & $\underline{B}$

B. Consistent Models:

1. Kinetic Description:
   a. Describe positions and velocities of all particles $\Rightarrow$ $f(\underline{x},\underline{v},t)$ Distribution function
   b. Klimontovich Equation:
      i. Exact description of every ion and electron in system
      ii. Together with Maxwell's Equation's, completely deterministic
      iii. Too detailed for practical application
   c. Liouville Equation: Another exact description of system $\Rightarrow$ too detailed.
   d. Plasma Kinetic Equation (Boltzmann Equation)
      i. Statistical treatment of a plasma
      ii. Evolves six-dimensional distribution function $f(\underline{x},\underline{v},t)$ due to $\underline{E}$ & $\underline{B}$ fields and collisions.
      iii. Vlasov Equation: Limit of Plasma Kinetic Equation when collisions $\to 0$.

2. Fluid Description:
   a. Velocity moments of the distribution function $f(\underline{x},\underline{v},t)$ [integrating over velocity space] lead to fluid variables [functions of $(\underline{x},t)$ only].

Lecture #2 (Continued) Howes (2)

I. 2. (Continued)

b. Evolution for each moment involves a higher moment

$\Rightarrow$ This leads to a closure problem

c. A physically motivated approximation is used to close equations.

d. $\boxed{\text{Two Fluid Equations}}$

i. Ions and electrons are each evolved separately

e. $\boxed{\text{MHD Equations}}$

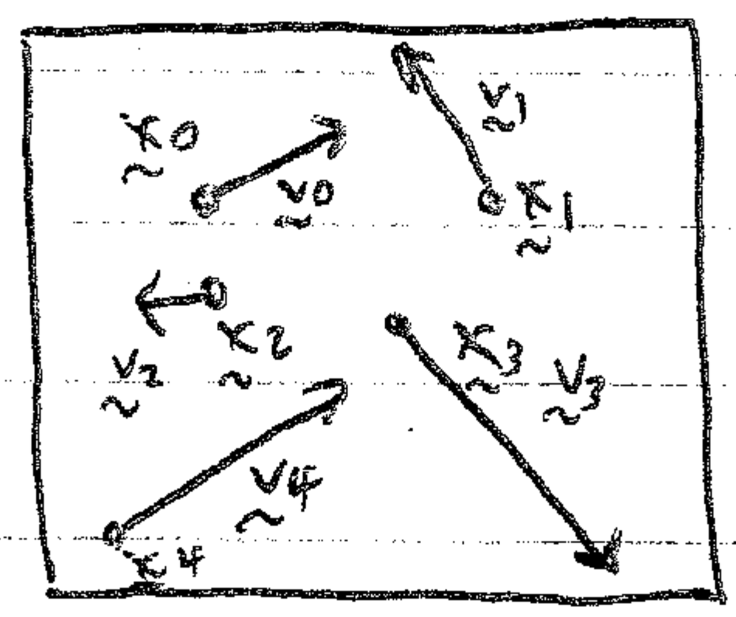
i. Ions and electrons move together, leading to a single fluid theory.

ii. Simple, consistent description of plasma behavior, but only valid when MHD approximation is satisfied.

II. Klimontovich Equation:

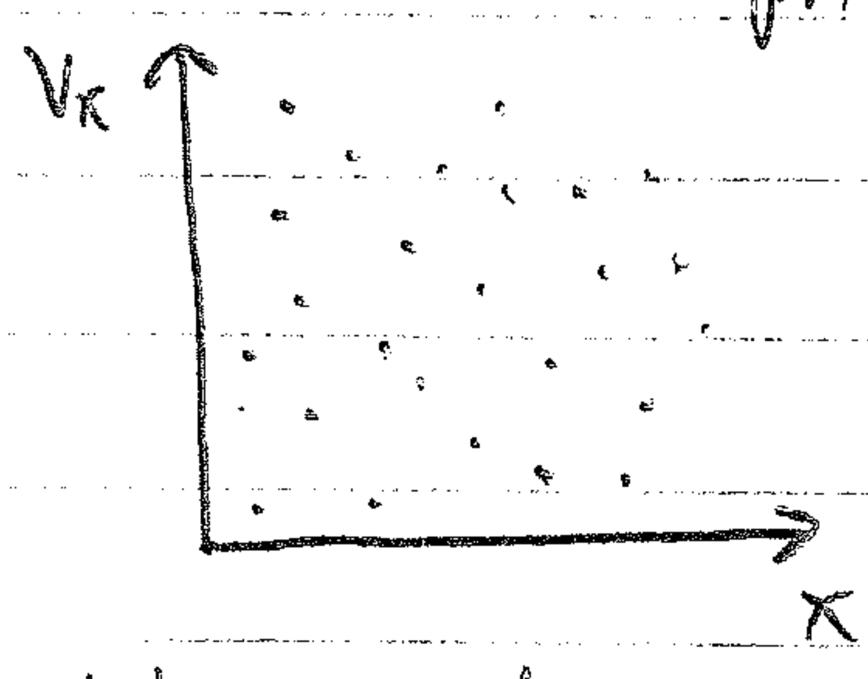
A. Phase Space:

1. A plasma is completely described if we know the position $\underset{\sim}{x}$ and velocity $\underset{\sim}{v}$ of each particle at time t.



a. We can introduce the six-dimensional phase space $(\underset{\sim}{x}, \underset{\sim}{v})$ such that each of the N particles occupies a position $(\underset{\sim}{x}_i, \underset{\sim}{v}_i)$ for $i=1,\dots,N$.

b. In this 6-D phase space, each particle occupies a point.



c. In a real plasma, the number of particles is huge, $\sim 10^{10}-10^{30}$ particles.

2. We can describe this N-particle plasma with the

a. Klimontovich Distribution

$$\boxed{F = \sum_{i=1}^{N} \delta[\underset{\sim}{x} - \underset{\sim}{x}_i(t)]\,\delta[\underset{\sim}{v} - \underset{\sim}{v}_i(t)]}$$

b. $\underset{\sim}{x}_i(t)$ = position of i^{th} particle

$\underset{\sim}{v}_i(t)$ = velocity of i^{th} particle

II. A2. (Continued)

c. $\frac{d\underset{\sim}{r}_i}{dt} = \underset{\sim}{v}_i$ $\frac{d\underset{\sim}{v}_i}{dt} = \underset{\sim}{a}_i = \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v}_i \times \underset{\sim}{B})$

d. $\mathcal{F}$ describes the density of particles in phase space. If we integrate over all velocities and position, we get number of particles

$$N = \int d^3\underset{\sim}{r} \int d^3\underset{\sim}{v}\ \mathcal{F}(\underset{\sim}{x}, \underset{\sim}{v}, t)$$

e. 1. $\mathcal{F}$ depends on detailed initial positions and velocities ~~of~~ each particle
2. $\mathcal{F}$ is far ~~too~~ detailed for practical use
3. This level of detail is not needed for most important macroscopic results.

f. How does $\mathcal{F}$ change in time? $\frac{d\mathcal{F}}{dt} = ?$
1. $\mathcal{F}(\underset{\sim}{x}, \underset{\sim}{v}, t)$ depends on 3 position, 3 velocity, and 1 time variables
2. $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} + \frac{dv_x}{dt}\frac{\partial}{\partial v_x} + \frac{dv_y}{dt}\frac{\partial}{\partial v_y} + \frac{dv_z}{dt}\frac{\partial}{\partial z}$
3. If we evaluate $\left.\frac{d\mathcal{F}}{dt}\right|_{orbit}$ along the particle orbits.
   i. $\left.\frac{dx}{dt}\right|_{orbit} = v_x$, $\left.\frac{dy}{dt}\right|_{orbit} = v_y$, etc. $\Rightarrow$ $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v}$
   ii. Similarly $\left.\frac{dv_x}{dt}\right|_{orbit} = a_x$, etc. $\Rightarrow$ $\frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{a}$
4. Thus $\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla + \underset{\sim}{a}\cdot\frac{\partial}{\partial \underset{\sim}{v}}}$

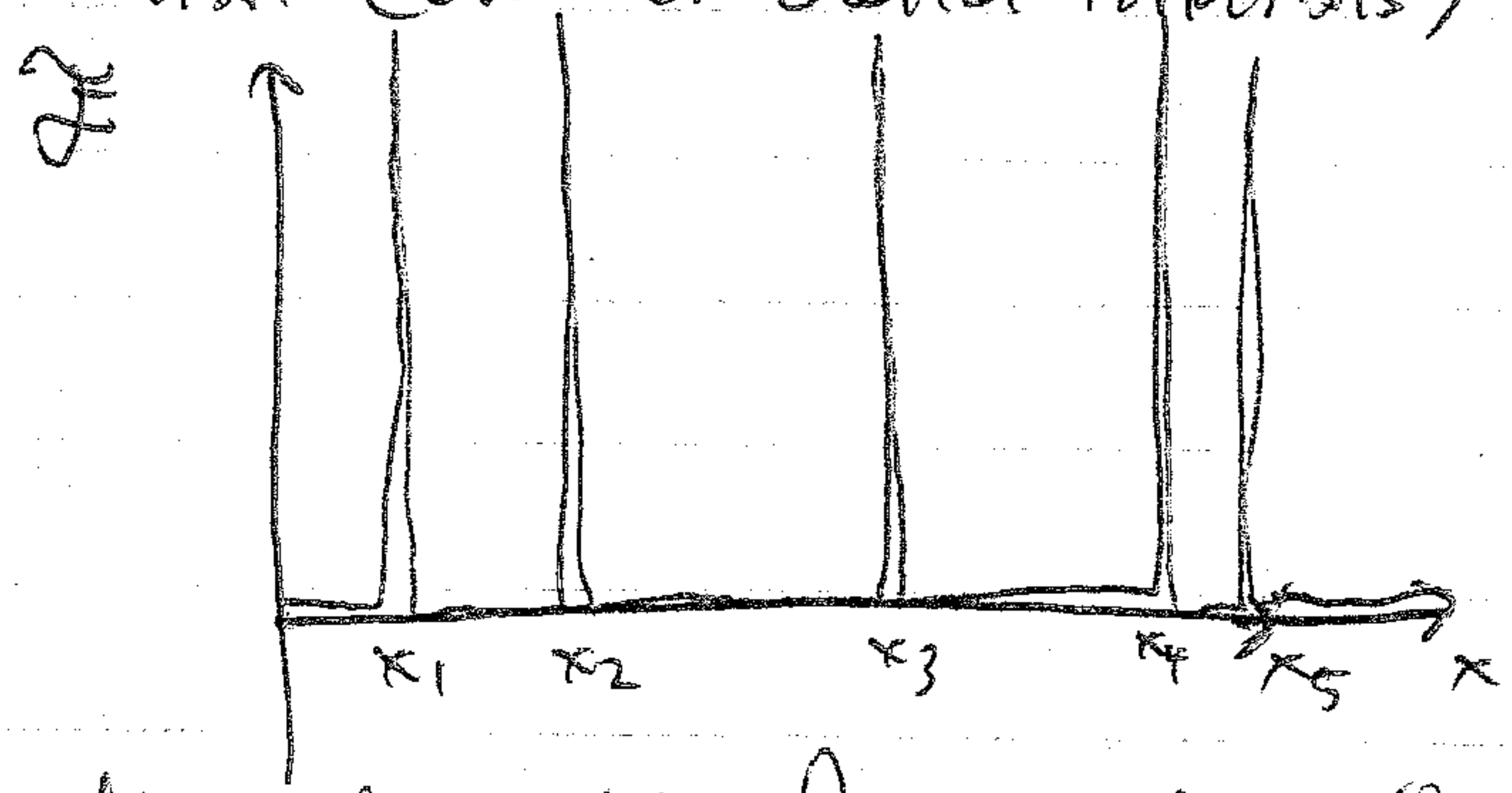
g. for a single species s, we get ~~the~~

$$\frac{d\mathcal{F}_s}{dt} = \boxed{\frac{\partial \mathcal{F}_s}{\partial t} + \underset{\sim}{v}\cdot\nabla\mathcal{F}_s + \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})\cdot\frac{\partial \mathcal{F}_s}{\partial \underset{\sim}{v}} = 0}$$

Klimontovich Equation

II. (Continued)

B. 1. The Klimontovich Distribution is a very spiky function (sum of delta functions) due to particle discreteness.



2. Along the orbit of particles, the distribution does not change $\frac{d\mathcal{F}}{dt} = 0$. Either you are on a particle, and density is "infinite"; or not on a particle, density is zero.

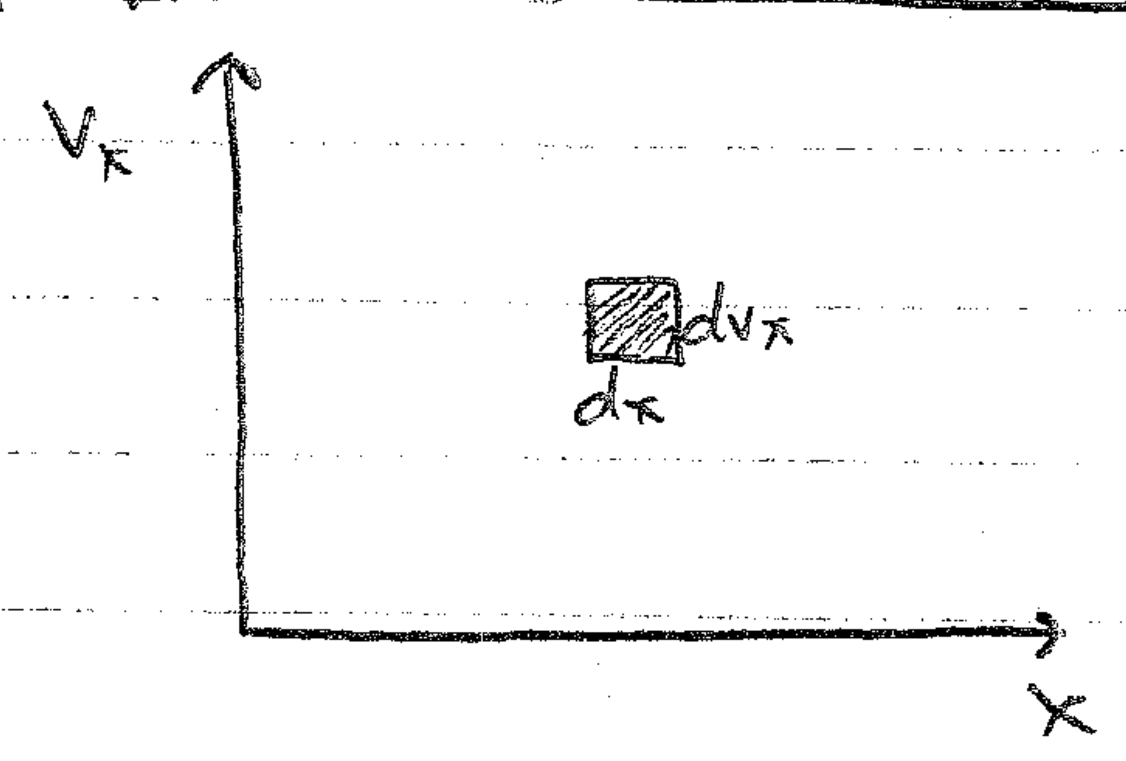
3. Evolving the Klimontovich Distribution is equivalent to an N-body problem.

a. For practical situations with $N \sim 10^{20}$, this is not possible

⇒ 4. We prefer a statistical description that ~~smoothly~~ averages over many particles, smoothing out the spiky-ness due to discrete particles, yielding a smooth solution.

III. Averaging to Yield Plasma Kinetic Equation

A. Coarse Grained Average:



1. Create a smoothed distribution by averaging over a small volume $d^3\mathbf{r}\, d^3\mathbf{v}$

2. This volume contains many particles, but is small enough that average doesn't change much over the volume.

3. For example, average with spherical exponential weighting

"Weight Function" $W(\mathbf{x}', \mathbf{v}') = e^{-\left(\frac{|\mathbf{x}'|}{x_0}\right)^2 - \left(\frac{|\mathbf{v}'|}{v_0}\right)^2}$ (characteristic sizes x_0, v_0.)

$x_0 \ll \lambda_D$ but $x_0 \gg n^{-1/3}$

III. A (Continued)

4.

a. $$\underbrace{f(\underset{\sim}{x}, \underset{\sim}{v}, t)}_{\text{Distribution Function}} = \int d^3\underset{\sim}{r}' \int d^3\underset{\sim}{v}' \, W(\underset{\sim}{x}', \underset{\sim}{v}') \, \mathcal{F}(\underset{\sim}{x}-\underset{\sim}{x}', \underset{\sim}{v}-\underset{\sim}{v}', t)$$

b. The distribution function $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$ is a statistical density in 6-D phase space over small volume $d^3\underset{\sim}{r}\, d^3\underset{\sim}{v}$

$\Rightarrow$ Smooths out "discreteness" of Klimontovich.

c. Enough particles in $d^3\underset{\sim}{r}\, d^3\underset{\sim}{v}$ to yield a good statistical average $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$.

5\. a. An alternative approach uses the Liouville Equation to describe a system of particles

b. An ensemble average of such systems leads to a similar statistical description of kinetic plasmas.

B. Separating Smooth and Fluctuating parts:

1\. a. The Klimontovich Equation describes the evolution of all plasma particles exactly.

b. We wish to find an evolution equation for the distribution function $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$

2\. Moments of Klimontovich Distribution: Charge and Current Densities.

a. Charge Density

$$\rho(\underset{\sim}{x}, t) = \sum_s \int d^3\underset{\sim}{v} \, q_s \, \mathcal{F}_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$$ "Zeroth" velocity moment

b. Current Density

$$\underset{\sim}{j}(\underset{\sim}{x}, t) = \sum_s \int d^3\underset{\sim}{v} \, q_s \, \underset{\sim}{v} \, \mathcal{F}_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$$ "First" velocity moment

Lecture #12 (Continued) Howes ⑥

III. B. (Continued)

3. Maxwell's Equations: ρ and $\underset{\sim}{j}$ are Sources for $\underset{\sim}{E}$ & $\underset{\sim}{B}$

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$

4. Split into Smoothed and Fluctuating parts.

a. Take $\mathcal{F}_s = \underbrace{f_s}_{\text{Smoothed part}} + \underbrace{(\mathcal{F}_s - f_s)}_{\text{Fluctuating due to particle discreteness}}$

b. Thus $\rho(\underset{\sim}{x},t) = \bar{\rho}(\underset{\sim}{x},t) + \tilde{\rho}(\underset{\sim}{x},t)$

where $\bar{\rho}(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\, q_s f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

$\tilde{\rho}(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\, q_s \left[\mathcal{F}_s(\underset{\sim}{x},\underset{\sim}{v},t) - f_s(\underset{\sim}{x},\underset{\sim}{v},t)\right]$

c. Similarly $\underset{\sim}{j}(\underset{\sim}{x},t) = \bar{\underset{\sim}{j}}(\underset{\sim}{x},t) + \tilde{\underset{\sim}{j}}(\underset{\sim}{x},t)$

5. Maxwell's Equations are linear, so we can perform the same separation for the fields $\underset{\sim}{E}$ & $\underset{\sim}{B}$

a. For example

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0} \Rightarrow \begin{cases} \nabla \cdot \bar{\underset{\sim}{E}} = \dfrac{\bar{\rho}}{\epsilon_0} \\ \nabla \cdot \tilde{\underset{\sim}{E}} = \dfrac{\tilde{\rho}}{\epsilon_0} \end{cases}$$

and the same for $\underset{\sim}{B}$.

C. Averaging the Klimontovich Equation:

1. Now, we perform the same averaging procedure on $\frac{d\mathcal{F}}{dt} = 0$ to yield.

$$\boxed{\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\bar{\underset{\sim}{E}} + \underset{\sim}{v} \times \bar{\underset{\sim}{B}}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = \underbrace{\frac{q_s}{m_s}\left\langle \left(\tilde{\underset{\sim}{E}} + \underset{\sim}{v} \times \tilde{\underset{\sim}{B}}\right) \cdot \frac{\partial \mathcal{F}_s}{\partial \underset{\sim}{v}} \right\rangle}_{= \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}}}$$

Plasma Kinetic Equation (or Boltzmann Equation)

Lecture #12 (Continued)

III C. (Continued)

2. a. The LHS contains only terms that vary smoothly in $(\underset{\sim}{x}, \underset{\sim}{v})$ space, i.e. f_s, $\underset{\sim}{E}$, $\underset{\sim}{B}$.

$\Rightarrow$ Collective effects of plasma

b. The RHS contains very spiky quantities due to particle discreteness. $\Rightarrow$ Collisional effects of plasma

3. Recall that ratio of $\frac{\text{Collective effects}}{\text{Collisional effects}} \sim \frac{\omega_{pe}}{\nu_c} \sim N_D$

Thus, collisional effects are typically weak.

4. We can neglect the effect of collisions, a good approximation for most plasma, to give

$$\boxed{\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0}$$

Vlasov Equation

$$\nabla \cdot \underset{\sim}{E} = \frac{1}{\epsilon_0} \overbrace{\sum_s \int d^3\underset{\sim}{v}\, q_s\, f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)}^{=\rho}$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \overbrace{\sum_s \int d^3\underset{\sim}{v}\, q_s v_s f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)}^{=\underset{\sim}{j}} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

Maxwell's Equations

These $\boxed{\text{Vlasov-Maxwell Equations}}$ describe the <u>collisionless</u> kinetic evolution of a plasma.

"Integro-differential" system of equations.

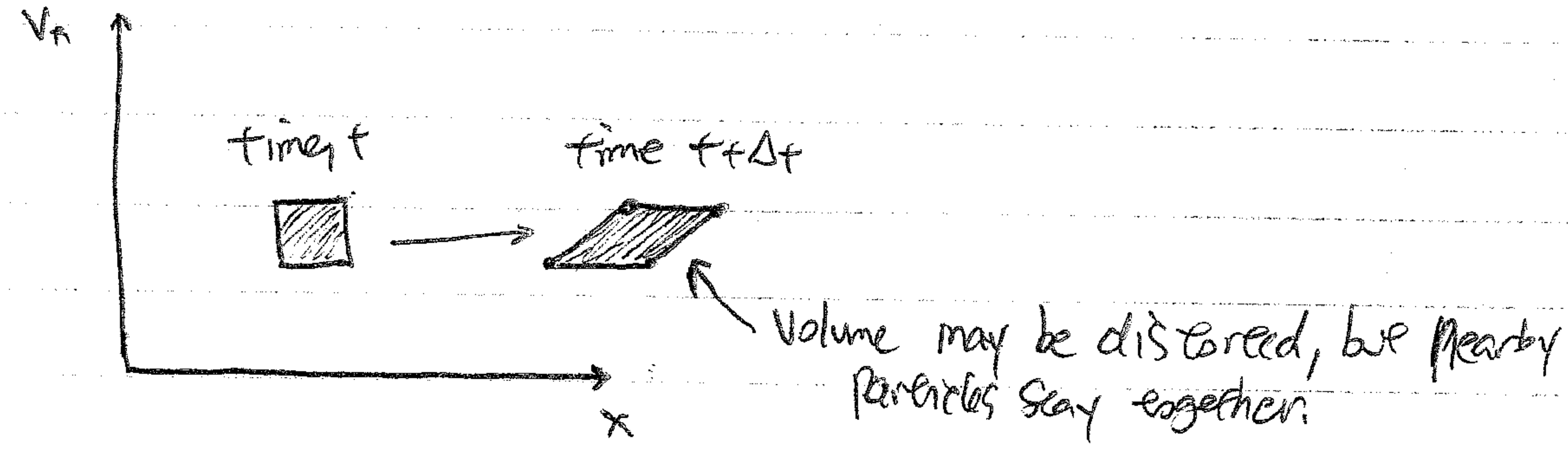
IV. The Distribution Function $f_s(\mathbf{x}, \mathbf{v}, t)$

A. Intuitive Picture:

1. $\int d^3\mathbf{x}\, d^3\mathbf{v}\, f_s(\mathbf{x}, \mathbf{v}, t)$ is the number of particles of species s in a infinitesimal volume in 6-D phase space, $\Delta v_x \Delta v_y \Delta v_z \Delta x \Delta y \Delta z$.

Thus $f_s(\mathbf{x}, \mathbf{v}, t) = \dfrac{\text{Number}}{\text{Volume}}$ is a number density in 6-D phase space.

2.

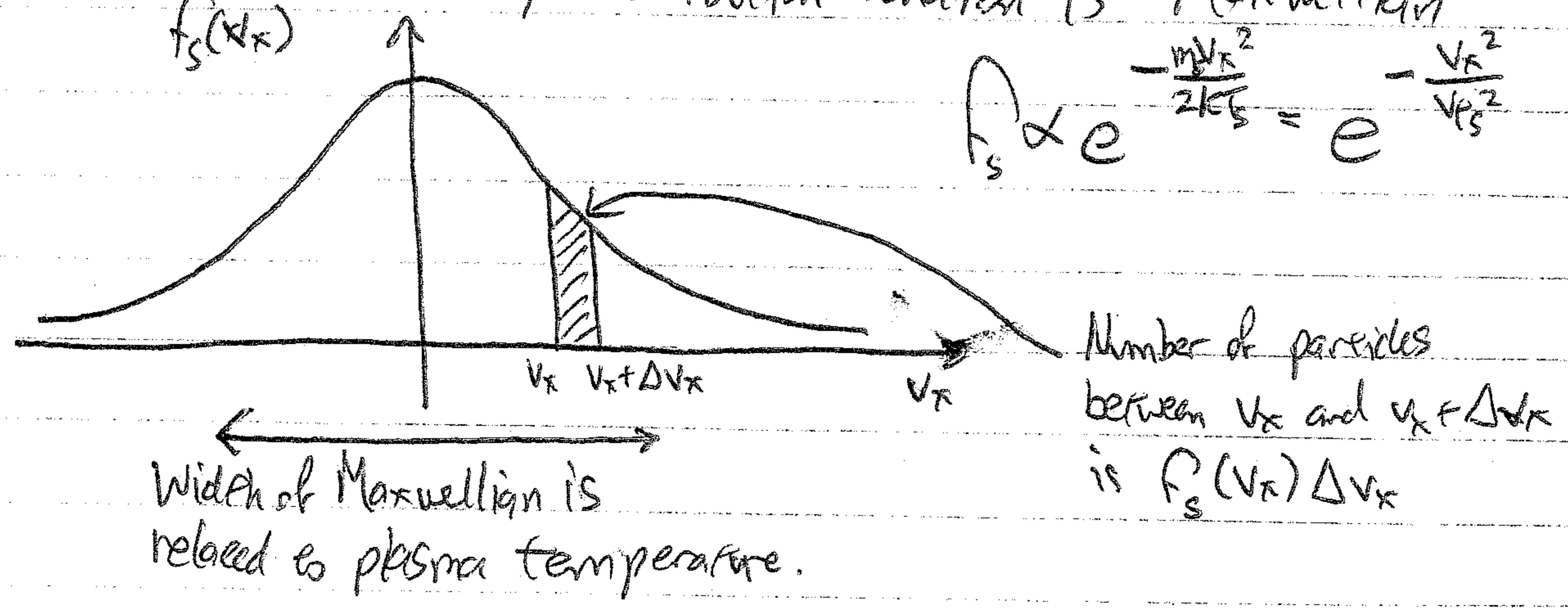


a. Particles at nearby points in phase space move together along their trajectories

b. $\dfrac{df_s}{dt} = 0$ (collisionless) suggests the distribution function is the density of an incompressible, 6-D fluid.

3. In Thermal Equilibrium, distribution function is Maxwellian

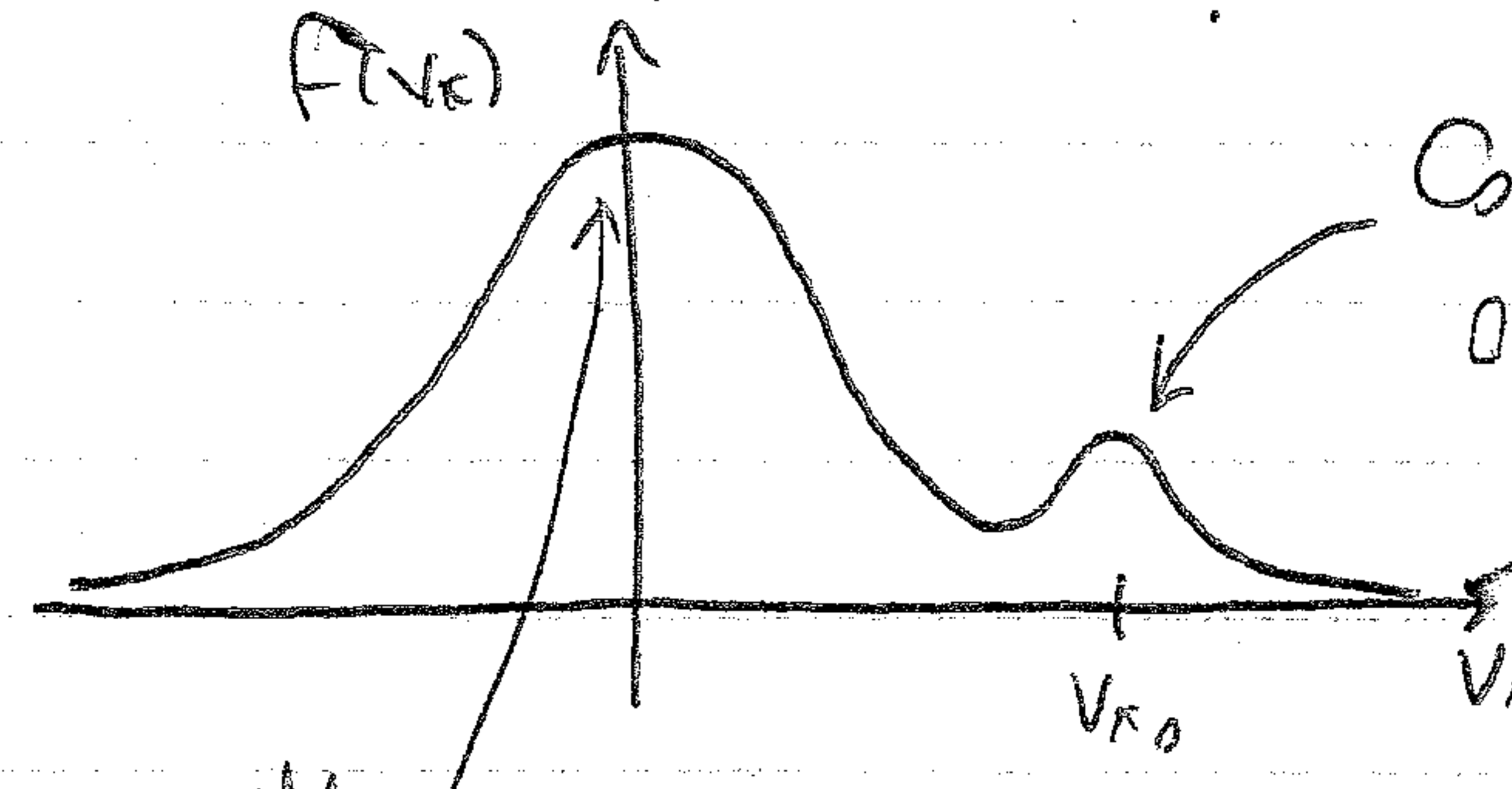
$$f_s \propto e^{-\frac{m v_x^2}{2kT_s}} = e^{-\frac{v_x^2}{v_{ts}^2}}$$



Lecture #12 (Continued)

IV. A. (Continued)

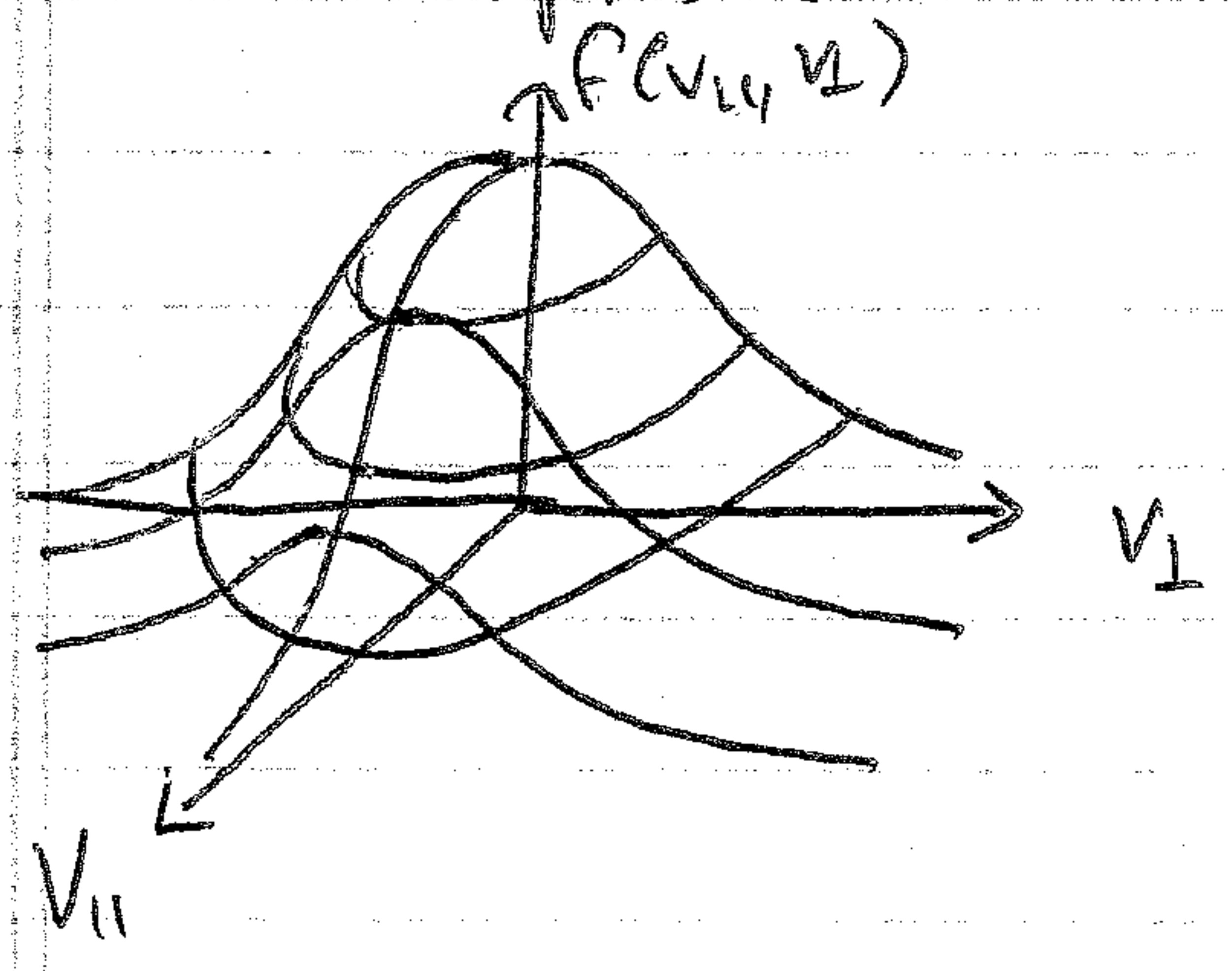
4. Non-Maxwellian Distribution Function

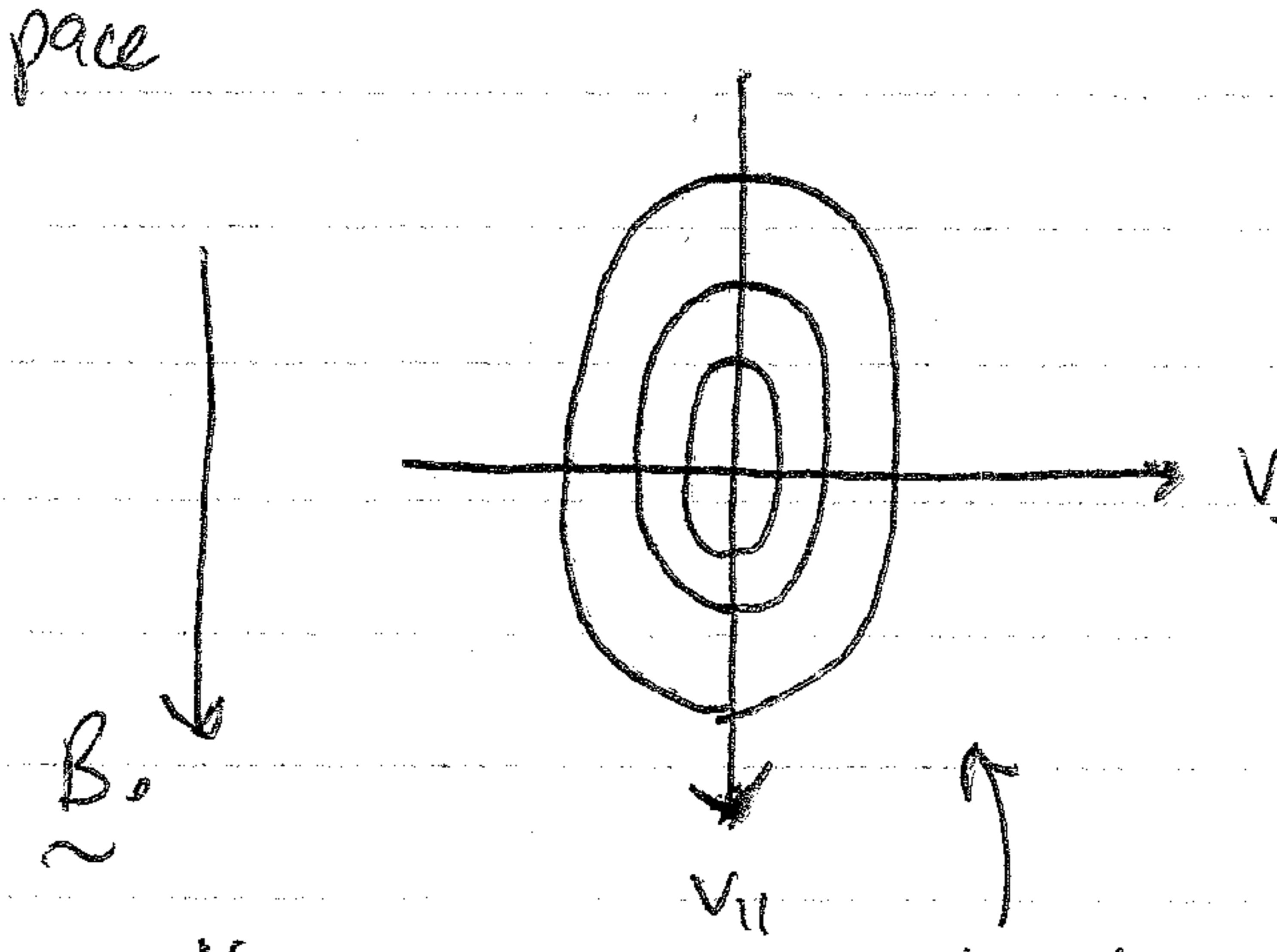


Corresponds to an added "beam" of particles moving at speed v_{x0} with respect to Maxwellian plasma at rest

Maxwellian plasma at rest

5. Contour plots of velocity space





Here, temperature in the direction parallel to $\mathbf{B}_0$ is greater than perpendicular temp.

B. Moments of the distribution function:

1. Density: $n_s(\mathbf{x},t) = \int d^3\mathbf{v}\, f_s(\mathbf{x},\mathbf{v},t) = \dfrac{\text{Number of particles}}{\text{unit volume}}$

2. Fluid Velocity: $\mathbf{U}_s(\mathbf{x},t) = \dfrac{\int d^3\mathbf{v}\; \mathbf{v}\, f_s(\mathbf{x},\mathbf{v},t)}{n_s(\mathbf{x},t)}$

3. Kinetic Energy Density: $\mathcal{E}(\mathbf{x},t) = \int d^3\mathbf{v}\, \frac{1}{2} m|\mathbf{v}|^2 f_s(\mathbf{x},\mathbf{v},t)$

4. Pressure Tensor: $\mathbf{P}_s(\mathbf{x},t) = \int d^3\mathbf{v}\, (\mathbf{v}-\mathbf{U}_s)(\mathbf{v}-\mathbf{U}_s)\, f_s(\mathbf{x},\mathbf{v},t)$

Next time, we'll use velocity moments to derive fluid equations.