Lecture #13 Moment Equations

Howes (1)

I. Review of Kinetic Description of Plasmas

A. Picture of a Kinetic Plasma.

1. At any point in space, a <u>kinetic</u> plasma contains particles moving with a distribution of velocities

2. The <u>distribution function</u> $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ describes a statistical measure of the plasma in a volume of ~~scale~~ x_0^3 such that

$x_0 \gg n^{-\frac{1}{3}}$ larger than particle spacing $\Rightarrow$ many particles

$x_0 \ll \lambda_D$ smaller than Debye length

3. $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ is a number density in 6-D phase space $(\underset{\sim}{x}, \underset{\sim}{v})$

B. Evolution of distribution function

1. Along particle orbit, $\dfrac{df_s}{dt} = \dfrac{\partial f_s}{\partial t} + \dfrac{d\underset{\sim}{x}}{dt} \cdot \dfrac{\partial f_s}{\partial \underset{\sim}{x}} + \dfrac{\partial \underset{\sim}{v}}{\partial t} \cdot \dfrac{\partial f_s}{\partial \underset{\sim}{v}}$

2. The statistical ensemble average yields

<u>Plasma Kinetic Equation</u>

$$\underbrace{\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}}}_{\text{Statistically smooth description of particles } f_s \text{, and fields } \underset{\sim}{E} \,\&\, \underset{\sim}{B} \text{ over volume } x_0^3} = \underbrace{\left(\frac{\partial f_s}{\partial t}\right)_{coll}}_{\text{Effect of discrete particles on smaller scales} \Rightarrow \text{Collisions.}}$$

3. Neglect the effect of collisions yields <u>Vlasov Equation</u>

$$\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0$$

~~7(Continued)~~
I. (Continued)

C. Moments of the Distribution Function

1. Summing over velocity space gives useful macroscopic quantities.

a. Density $n_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\ f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

b. Fluid Velocity $\underset{\sim}{U}_s(\underset{\sim}{x},t) = \frac{1}{n_s(\underset{\sim}{x},t)} \int d^3\underset{\sim}{v}\ \underset{\sim}{v}\ f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

c. Kinetic Energy Density: $\mathcal{E}_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\ \frac{1}{2} m |\underset{\sim}{v}|^2 f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

d. Pressure Tensor: $\underset{\approx}{P}_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\ m_s(\underset{\sim}{v}-\underset{\sim}{U}_s)(\underset{\sim}{v}-\underset{\sim}{U}_s) f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

2. These are the velocity moments of the distribution function.

3. Taking moments of the kinetic distribution function gives "fluid" quantities, or observable quantities.

a. ~~Ex:~~ The temperature of a kinetic species s can be defined,

$$\mathcal{E}_s \equiv \frac{3}{2} n_s k T_s \quad \Rightarrow T_s \equiv \frac{2\,\mathcal{E}_s}{3 n_s k}$$

II. From a Kinetic to a Fluid Description

A. Fluid Description:

1. Often, we don't care about detailed distribution of velocities.
2. Thus, all we want are velocity moments, n_s, $\underset{\sim}{U}_s$, $\mathcal{E}_s$, etc.
3. The velocity moments just define the dependent variables of a fluid description, which depend only on $\underset{\sim}{x}$ and t.

II. (Continued)

B. Comparison of Kinetic vs. Fluid Descriptions

Kinetic

1. $\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0$

Charge Density: $\rho = \sum_s q_s \int d^3v \, f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$

Current Density: $\underset{\sim}{j} = \sum_s q_s \int d^3v \, \underset{\sim}{v} \, f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$

$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0}$ $\qquad \nabla \cdot B = 0$

$\nabla \times E = -\frac{\partial B}{\partial t}$ $\qquad \nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

2. Everything depends on 6-D distribution function $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$

Fluid:

1. Moments: $n_s(\underset{\sim}{x}, t)$
   $\underset{\sim}{U}_s(\underset{\sim}{x}, t)$
   $\underset{\sim}{\underset{\sim}{P}}_s(\underset{\sim}{x}, t)$
   etc.

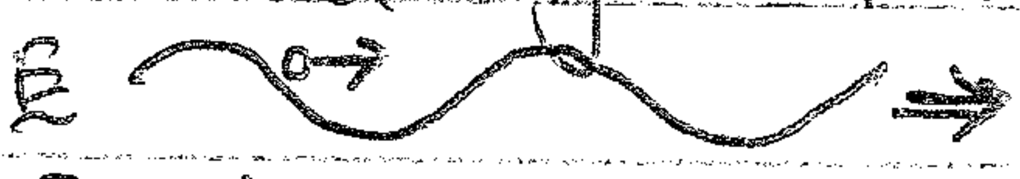
2. Evolution equations (to be derived next)

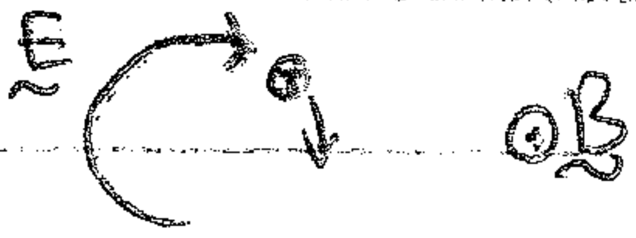
3. A number of fluid variables (moments), all of which <u>only</u> depend on $\underset{\sim}{x}$ and t.

C. <u>When is a fluid description appropriate?</u>

<u>ANSWER</u>: When the details of the velocity distribution do not significantly affect the evolution.

1. <u>Ex</u>: Fluid description is not valid when particles at certain velocity become resonant with fluctuating EM fields.

$\Rightarrow$ Landau Damping 

$\Rightarrow$ Cyclotron Damping 

2. <u>Ex</u>: Fluid description is appropriate for strongly collisional plasmas. Collisions drive towards <u>Local Thermodynamic Equilibrium</u>

$\Rightarrow$ Velocity distribution becomes <u>Maxwellian</u>.

II. (Continued)

D. Maxwellian (and Bi-Maxwellian) Plasmas:

1. Fluid equations are a ~~[illegible]~~ good description of a plasma when the velocity distributions remain Maxwellian

2. Maxwellian Distribution
$$\boxed{f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = \frac{n_s(\underset{\sim}{x},t)}{\pi^{3/2} v_{ts}^3} e^{-\frac{m_s|\underset{\sim}{v}-\underset{\sim}{U}(\underset{\sim}{x},t)|^2}{2kT_s(\underset{\sim}{x},t)}}}$$

a. Fluid is completely described by three fluid variables:

Density $n_s(\underset{\sim}{x},t)$

Fluid Velocity $\underset{\sim}{U}_s(\underset{\sim}{x},t)$

Temperature $T_s(\underset{\sim}{x},t)$

b. Here, as usual, the thermal velocity $v_{ts} = \sqrt{\frac{2kT_s}{m_s}}$

3. Zeroth Moment: Density

a. $$\int d^3\underset{\sim}{v}\, f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = \int_{-\infty}^{\infty} dv_x\, dv_y\, dv_z \frac{n_s}{\pi^{3/2} v_{ts}^3} e^{-\frac{|\underset{\sim}{v}-\underset{\sim}{U}|^2}{v_{ts}^2}}$$

b. NOTE: $|\underset{\sim}{v}-\underset{\sim}{U}|^2 = (v_x-U_x)^2+(v_y-U_y)^2+(v_z-U_z)^2$

c. We can split this into three integrals which are identical

$$\frac{1}{\pi^{1/2} v_{ts}} \int_{-\infty}^{\infty} dv_x\, e^{-\frac{(v_x-U_x)^2}{v_{ts}^2}} = \frac{1}{\pi^{1/2} v_{ts}} \underbrace{\int_{-\infty}^{\infty} v_{ts}\, dy\, e^{-y^2}}_{=v_{ts}\sqrt{\pi}} = \frac{\sqrt{\pi}\, \cancel{v_{ts}}}{\pi^{1/2}\, \cancel{v_{ts}}} = 1$$

$y = \frac{v_x-U_x}{v_{ts}}$ $dy = \frac{dv_x}{v_{ts}}$

d. Thus $$\boxed{\int d^3\underset{\sim}{v}\, f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = n_s(\underset{\sim}{x},t)}$$

4. First Moment: Fluid Velocity

$$\int d^3\underset{\sim}{v}\, \underset{\sim}{v} f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = n_s(\underset{\sim}{x},t)\, \underset{\sim}{U}_s(\underset{\sim}{x},t)$$

5. Second Moment: Energy

$$\int d^3\underset{\sim}{v}\, \frac{1}{2} m_s |\underset{\sim}{v}|^2 f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = \underbrace{\frac{3}{2} n_s k T_s}_{\text{Thermal Energy}} + \underbrace{\frac{1}{2} m_s n_s |\underset{\sim}{U}_s|^2}_{\text{Kinetic Energy of Fluid Flow}}$$

6. The presence of a Mean Magnetic Field $\underset{\sim}{B}_0$ often leads to an anisotropic distribution function

Bi-Maxwellian Distribution: $\boxed{f_{s_{Bm}}(\underset{\sim}{x},\underset{\sim}{v},t) = \frac{n_s(\underset{\sim}{x},t)}{\pi^{3/2} v_{t_{\perp s}} v_{t_{\parallel s}}^{\frac{1}{2}}} e^{-\frac{m v_\perp^2}{2kT_{s\perp}} - \frac{m v_\parallel^2}{2kT_{s\parallel}}}}$

where

$$v_{t_{\perp s}} = \sqrt{\frac{2kT_{s\perp}}{m_s}} \quad \text{and} \quad v_{t_{\parallel s}} = \sqrt{\frac{2kT_{s\parallel}}{m_s}}$$

a. This can lead to the $\boxed{\text{Chew–Goldberger–Low, or Double Adiabatic, Equation of State}}$

III. Moment Equations:

Moments of the Plasma Kinetic Equation determine evolution of velocity moments.

A. Zeroth Moment: Notation $\int_{\underset{\sim}{v}} = \int d^3\underset{\sim}{v}$

$$\underbrace{\int_{\underset{\sim}{v}} \frac{\partial f_s}{\partial t}}_{①} + \underbrace{\int_{\underset{\sim}{v}} \underset{\sim}{v}\cdot\nabla f_s}_{②} + \underbrace{\int_{\underset{\sim}{v}} \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_s}_{③} = \underbrace{\int_{\underset{\sim}{v}} \left(\frac{\partial f_s}{\partial t}\right)_{coll}}_{④}$$

1. $① = \frac{\partial}{\partial t}\int_{\underset{\sim}{v}} f_s = \frac{\partial}{\partial t} n_s$

Lecture #1B (Continued) Howes (6)

III. A. (Continued)

2. ② $= \int_{\underset{\sim}{v}} \underset{\sim}{v} \cdot \nabla f_s = \nabla \cdot \int_{\underset{\sim}{v}} \underset{\sim}{v} f_s = \nabla \cdot (n_s \underset{\sim}{U_s})$

a. NOTE: $\underset{\sim}{x}$ and $\underset{\sim}{v}$ are independent variables, so $\nabla \cdot (\underset{\sim}{v} f_s) = \underset{\sim}{v} \cdot \nabla f_s + f_s \cancelto{0}{\nabla \cdot \underset{\sim}{v}}$

3. ③ $= \frac{q_s}{m_s} \int_{\underset{\sim}{v}} (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = \frac{q_s}{m_s} \int_{\underset{\sim}{v}} \nabla_{\underset{\sim}{v}} \cdot [(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) f_s]$

a. NOTE: $\underset{\sim}{E}(\underset{\sim}{x},t)$ & $\underset{\sim}{B}(\underset{\sim}{x},t)$ are independent of $\underset{\sim}{v}$: $\frac{\partial}{\partial \underset{\sim}{v}} \cdot (\underset{\sim}{E} f_s) = \underset{\sim}{E} \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} + f_s \cancelto{0}{\frac{\partial}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{E}}$

b. $\frac{\partial}{\partial \underset{\sim}{v}} \cdot ((\underset{\sim}{v} \times \underset{\sim}{B}) f_s) = \underset{\sim}{v} \times \underset{\sim}{B} \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} + f_s \cancelto{0}{\frac{\partial}{\partial \underset{\sim}{v}} \cdot (\underset{\sim}{v} \times \underset{\sim}{B})}$

Since each component is like $\frac{\partial}{\partial v_x} (\underset{\sim}{v} \times \underset{\sim}{B})_x = \frac{\partial}{\partial v_x} (v_y B_z - v_z B_y) = 0$

c. Using NRL p.5 (28) Gauss's Thm: $\int_V d^3\underset{\sim}{x} \, \nabla \cdot \underset{\sim}{A} = \int_S d\underset{\sim}{S} \cdot \underset{\sim}{A}$

$\Rightarrow$ ③ $= \frac{q_s}{m_s} \int d\underset{\sim}{S}_v \cdot [(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot f_s] = 0$

d. At $v \to \pm\infty$, $f_s \to 0$, so surface integral vanishes at infinity.

4. ④ $= \left(\frac{\partial}{\partial t} \int_{\underset{\sim}{v}} f_s \right)_{coll} = \left(\frac{\partial n_s}{\partial t} \right)_{coll} = 0$

Collisions don't create or destroy particles (number is conserved).

5. Thus, we are left with

$$\boxed{\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underset{\sim}{U_s}) = 0}$$ Continuity Equation

a. Expresses conservation of particles

b. Integrate over a small volume V: $\int_V d^3\underset{\sim}{x} \frac{\partial n_s}{\partial t} = - \int_V d^3\underset{\sim}{x} \, \nabla \cdot (n_s \underset{\sim}{U_s})$

i) Once again, use Gauss's Thm on RHS.

ii) N = number of particles in V, $\Rightarrow$ $\frac{\partial N}{\partial t} = - \int_S d\underset{\sim}{S} \cdot (n_s \underset{\sim}{U_s})$ Particles moving through surface

Lecture #13 (Continued)

III. B. First Moment: $\int_{\mathbf{v}} \mathbf{v}$ on Kinetic Equation

$$\underbrace{\int_{\mathbf{v}} \mathbf{v} \frac{\partial f_s}{\partial t}}_{①} + \underbrace{\int_{\mathbf{v}} \mathbf{v}\, \mathbf{v}\cdot\nabla f_s}_{②} + \underbrace{\int_{\mathbf{v}} \mathbf{v} \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_s}{\partial \mathbf{v}}}_{③} = \underbrace{\int_{\mathbf{v}} \mathbf{v}\left(\frac{\partial f_s}{\partial t}\right)_{coll}}_{④}$$

1. ① $= \frac{\partial}{\partial t}\int_{\mathbf{v}} \mathbf{v} f_s = \frac{\partial}{\partial t}(n_s \mathbf{U}_s)$

2. This term involves Tensor mathematics.

② $= \int_{\mathbf{v}} \mathbf{v}\mathbf{v}\cdot\nabla f_s = \nabla\cdot\left(\int_{\mathbf{v}} \mathbf{v}\mathbf{v} f_s\right)$ since $\mathbf{x}$ & $\mathbf{v}$ are independent.

a. Use $\mathbf{v} = \mathbf{U}_s + \mathbf{v} - \mathbf{U}_s$ to get $\mathbf{v}\mathbf{v} = \underbrace{\mathbf{U}_s\mathbf{U}_s}_{ⓐ} + \underbrace{\mathbf{U}_s(\mathbf{v}-\mathbf{U}_s)}_{ⓑ} + \underbrace{(\mathbf{v}-\mathbf{U}_s)\mathbf{U}_s}_{ⓒ} + \underbrace{(\mathbf{v}-\mathbf{U}_s)(\mathbf{v}-\mathbf{U}_s)}_{ⓓ}$

b. ⓐ $= \int_{\mathbf{v}} \mathbf{U}_s\mathbf{U}_s f_s = n_s \mathbf{U}_s\mathbf{U}_s$

c. ⓑ $= \int_{\mathbf{v}} \mathbf{U}_s(\mathbf{v}-\mathbf{U}_s) f_s = \mathbf{U}_s \int \mathbf{v} f_s - \mathbf{U}_s\mathbf{U}_s n_s = \mathbf{U}_s\mathbf{U}_s n_s - \mathbf{U}_s\mathbf{U}_s n_s = 0$

d. ⓒ same as ⓑ $= 0$

e. ⓓ Using $\mathbf{P}_s = \int_{\mathbf{v}} m_s (\mathbf{v}-\mathbf{U}_s)(\mathbf{v}-\mathbf{U}_s) f_s$, ⓓ $= \frac{\mathbf{P}_s}{m}$

f. Thus ② $= \nabla\cdot(n_s \mathbf{U}_s\mathbf{U}_s) + \frac{1}{m_s}\nabla\cdot\mathbf{P}_s$

3. ③ $= \frac{q_s}{m_s}\int_{\mathbf{v}} \mathbf{v}\frac{\partial}{\partial \mathbf{v}}\cdot\left[(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_s\right]$ Using same trick as Zeroth Moment.

a. Nine integrals for each pairing of v_x, v_y, v_z & $\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}$

i) Six have $v_i \frac{\partial}{\partial v_j}$ where $i \neq j$, thus are similar to

$$\int dv_x \int dv_y\, v_y \int dv_z \frac{\partial}{\partial v_z}\left[(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_s\right]$$

ii) v_z integral gives $(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_s\Big|_{-\infty}^{\infty} = 0$ as $v\to\infty$ since $f_s \to 0$.

iii) Three have $v_i \frac{\partial}{\partial v_i} \Rightarrow \int dv_x \int dv_y \int dv_z\, v_z \frac{\partial}{\partial v_z}\left[(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_s\right]$

III. B3. (Continued)

$$\int dv_z\, v_z \frac{\partial}{\partial v_z}\left[(\mathbf{E}+\mathbf{v}\times\mathbf{B}) f_s\right] = \cancel{v_z(\mathbf{E}+\mathbf{v}\times\mathbf{B}) f_s\Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) f_s\, dv_z$$

$u = v_z \qquad dv = \frac{\partial}{\partial v_z}\left[(\mathbf{E}+\mathbf{v}\times\mathbf{B}) f_s\right] dv_z$

$du = dv_z \qquad v = (\mathbf{E}+\mathbf{v}\times\mathbf{B}) f_s$

b. We get ③ $= -\frac{q_s}{m_s}(\mathbf{E}+\mathbf{U}_s\times\mathbf{B})$

4. ④ $= \frac{1}{m_s}\int_{\mathbf{v}} m_s \mathbf{v}\left(\frac{\partial f_s}{\partial t}\right)_{coll} = \frac{1}{m_s}\left(\frac{\partial}{\partial t}\int_{\mathbf{v}} m_s \mathbf{v} f_s\right)_{coll} = \frac{1}{m_s}\left(\frac{\partial\, n_s m_s \mathbf{U}_s}{\partial t}\right)_{coll}$

a. This represents a DRAG FORCE due to collisions between species

b. Same species collisions conserve momentum, so produce no drag.

5. Thus,

$$\frac{\partial}{\partial t}(n_s\mathbf{U}_s) + \nabla\cdot(n_s\mathbf{U}_s\mathbf{U}_s) = -\frac{1}{m_s}\nabla\cdot\mathbf{P}_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{U}_s\times\mathbf{B}) + \text{Collisional Drag}\,(\mathbf{F}_{Ds})$$

a. NOTE: LHS can be written, using tensor identity $\nabla\cdot(\mathbf{A}\mathbf{B}) = (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{A}\cdot\nabla)\mathbf{B}$

$$n_s\frac{\partial \mathbf{U}_s}{\partial t} + \underbrace{\mathbf{U}_s\frac{\partial n_s}{\partial t} + \mathbf{U}_s\nabla\cdot(n_s\mathbf{U}_s)}_{=\mathbf{U}_s\left[\cancel{\frac{\partial n_s}{\partial t}+\nabla\cdot(n_s\mathbf{U}_s)}^{\,0}\right]} + n_s\mathbf{U}_s\cdot\nabla\mathbf{U}_s$$

Continuity eq.

6. Multiplying by m_s, we get Momentum Equation

$$\boxed{n_s m_s\frac{\partial \mathbf{U}_s}{\partial t} + n_s m_s\mathbf{U}_s\cdot\nabla\mathbf{U}_s = -\nabla\cdot\mathbf{P}_s + q_s(\mathbf{E}+\mathbf{U}_s\times\mathbf{B}) + \mathbf{F}_{Ds}}$$

C. Second Moment: Taking $\int_{\mathbf{v}}\frac{1}{2}m_s|\mathbf{v}|^2$ of Kinetic Equation gives,

$$\boxed{\frac{\partial \mathcal{E}_s}{\partial t} + \nabla\cdot\mathbf{Q}_s - \mathbf{E}\cdot\mathbf{j}_s = \left(\frac{\partial\mathcal{E}_s}{\partial t}\right)_{coll}}$$

Energy Equation

Lecture #13 (Continued) Howes (9)

II. C. (Continued)

1. Here $\underset{\sim}{Q}_s$ is a heat flux

Heat Flux $\boxed{\underset{\sim}{Q}_s \equiv \int d^3\underset{\sim}{v}\ \frac{1}{2} m_s |\underset{\sim}{v}|^2 \underset{\sim}{v}\ f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)}$ (Third Velocity Moment)

2. $\underset{\sim}{E} \cdot \underset{\sim}{j}_s$ represents Joule heating of species s

3. $\left(\frac{\partial \mathcal{E}_s}{\partial t}\right)_{coll}$ is collisional heating due to collisions with other species.

a. Same-species conserve energy within species.

D. Closure Problem:

1. The evolution equation for the n^{th} moment involves the $(n+1)^{st}$ moment.

a. $$\underbrace{\frac{\partial n_s}{\partial t}}_{\text{(zeroth)}} + \nabla \cdot (\underbrace{n_s \underset{\sim}{U}_s}_{\text{(first)}}) = 0$$

b. $$n_s m_s \left[\underbrace{\frac{\partial \underset{\sim}{U}_s}{\partial t}}_{\text{(first)}} + \underset{\sim}{U}_s \cdot \nabla \underset{\sim}{U}_s\right] = -\nabla \cdot \underbrace{\underset{\approx}{P}_s}_{\text{(second)}} + q_s(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) + \underset{\sim}{F}_{D_s}$$

2. For N moment equations, a closed system can only be specified if the $(N+1)^{st}$ moment is related to the first N moments.

Ex: We may specify an equation of state relate pressure in terms of density.

Adiabatic Equation of State [illegible] $P_s n_s^{-\gamma} = \text{constant}$

$$\Rightarrow \frac{d}{dt}\left(\frac{P}{n^{\gamma}}\right) = 0$$