Lecture #14 The Two Fluid Equations

I. Simplifications of the Pressure Tensor $\underset{\approx}{P}_s$

A. Review

1. By taking the first moment of the Plasma Kinetic Equation, we derive the Fluid Momentum Equation

$$n_s m_s \underbrace{\left[\frac{\partial \underset{\sim}{U}_s}{\partial t} + \underset{\sim}{U}_s \cdot \nabla \underset{\sim}{U}_s\right]} = -\nabla \cdot \underset{\approx}{P}_s + n_s q_s (\underset{\sim}{E} + \underset{\sim}{U}_s \times \underset{\sim}{B}) + \underset{\sim}{F}_{D_s}$$

Collisional Drag

a. Convective Derivative: $\dfrac{d\underset{\sim}{U}_s}{dt} \equiv \dfrac{\partial \underset{\sim}{U}_s}{\partial t} + \underset{\sim}{U}_s \cdot \nabla \underset{\sim}{U}_s$

The total derivative along the trajectory of a fluid element.

B. Isotropic Pressure

1. The pressure tensor includes both pressure and viscosity.

a. $$\underset{\approx}{P}_s = p_s \underset{\approx}{I} + \underset{\approx}{\Pi}_s = \begin{bmatrix} p_s & 0 & 0 \\ 0 & p_s & 0 \\ 0 & 0 & p_s \end{bmatrix} + \underset{\approx}{\Pi}_s$$

$p_s = \frac{1}{3}\mathrm{Tr}[\underset{\approx}{P}_s]$ ← Viscosity Tensor

2. In this case $-\nabla \cdot \underset{\approx}{P}_s = -\nabla p_s - \nabla \cdot \underset{\approx}{\Pi}_s$

3. The viscosity leads to the transport of momentum.

a. $\underset{\approx}{\Pi}_s \sim \mathcal{O}\left(\dfrac{\lambda_m}{L} p_s\right)$ where L is scale length of fluid variables $n_s(\underset{\sim}{x},t)$, $\underset{\sim}{U}_s(\underset{\sim}{x},t)$, $T_s(\underset{\sim}{x},t)$.

b. In a collisional plasma, $\lambda_m \ll L$, so the viscous term is small compared to the pressure term.

c. Thus, the viscosity term is often dropped in a collisional fluid plasma.

$$\Rightarrow -\nabla \cdot \underset{\approx}{P}_s \simeq -\nabla p_s$$

4. For a Maxwellian distribution $f_{s_M}(\underset{\sim}{x},\underset{\sim}{v},t)$, $\underset{\approx}{P}_s = \int d^3\underset{\sim}{v}\, m_s (\underset{\sim}{v}-\underset{\sim}{U}_s)(\underset{\sim}{v}-\underset{\sim}{U}_s) f_{s_M}(\underset{\sim}{x},\underset{\sim}{v},t)$

gives $\underset{\approx}{P}_s = \begin{bmatrix} n_s k T_s & 0 & 0 \\ 0 & n_s k T_s & 0 \\ 0 & 0 & n_s k T_s \end{bmatrix}$ Thus $\boxed{p_s = n_s k T_s}$

2. (Continued)

C. Anisotropic Distribution

1. For a Bi-Maxwellian distribution $f_{BM}(\underset{\sim}{x}, \underset{\sim}{v}, t)$, we can integrate over velocity space to obtain

$$\underset{\approx}{P}_s = \begin{bmatrix} P_{\perp s} & 0 & 0 \\ 0 & P_{\perp s} & 0 \\ 0 & 0 & P_{\parallel s} \end{bmatrix} \quad \text{where} \quad \begin{aligned} P_{\perp s} &= n_s k T_{\perp s} \\ P_{\parallel s} &= n_s k T_{\parallel s} \end{aligned}$$

II. Solutions to the Closure Problem:

A. Introduction

1. As stated at the end of Lecture #13, the ~~procedure~~ of taking the n^{th} moment of the Plasma Kinetic Equation also involves the $(n+1)^{st}$ moment due to the term $\underset{\sim}{v} \cdot \nabla f_s$.
2. To close the system of equations, we need to specify the $(n+1)^{st}$ moment in terms of the first n moments.
   a. Ex: If we can specify $\underset{\approx}{P}_s = f(n_s, \underset{\sim}{U}_s)$, then we can close the system of equations.
3. Generally, this can only be done in an ad hoc manner.
   a. Usually close the system with a physically motivated Equation of State

B. Cold Plasma Equation of State

1. If we assume the temperature $T_s \to 0$, then $\underset{\approx}{P}_s \to 0$.
2. Neglecting collisional drag term, we obtain

$$m_s \frac{d\underset{\sim}{U}_s}{dt} = q_s(\underset{\sim}{E} + \underset{\sim}{U}_s \times \underset{\sim}{B})$$

3. This is the same form as the Lorentz Force Law, but $U_s(\underset{\sim}{x}, t)$ is the fluid velocity and

$$\frac{d}{dt}\underset{\sim}{U}_s = \frac{\partial \underset{\sim}{U}_s}{\partial t} + \underbrace{\underset{\sim}{U}_s \cdot \nabla \underset{\sim}{U}_s}$$

Convective Derivative Introduces a nonlinear term.

II. (Continued)

C. Adiabatic Equation of State

1. From thermodynamics, if no heat flow during a compression (adiabatic motion), ~~the~~

$$pV^{\gamma} = \text{constant}$$

where V = volume of gas

γ = adiabatic index $= \frac{C_p}{C_V}$

2. For this to be valid:
   a. Motion must be rapid to prevent heat flow (adiabatic)
   b. Collisions must maintain Local Thermodynamic Equilibrium
3. Ex: Sound waves in a gas a typically adiabatic.
4. In a collisionless plasma, the adiabatic equation of state is often not strictly valid, but it is nevertheless commonly used.
5. For a collisional plasma, $\underline{\underline{P}}_s \rightarrow p_s \underline{\underline{I}}$ (isotropic pressure)
6. Adiabatic Equation of State $\boxed{\frac{d}{dt}\left(\frac{p_s}{n_s^{\gamma}}\right) = 0}$ ⇐ Expresses the Conservation of Entropy

where $p_s = n_s k T_s$

and $\gamma = \frac{d+2}{d}$ for a gas with d degrees of freedom.

   a. For a monatomic "gas" → a plasma of H^+ ions and e^- electrons,
   $\gamma = \frac{5}{3}$.

D. Double Adiabatic Equation of State

Also known as Chew-Goldberger-Low (CGL) Equation of State.

1. In a magnetized plasma, the timescale to transfer momentum from parallel to perpendicular directions (and vice-versa) is often long.

⇒ Bi-Maxwellian distribution $f_{sBM}(T_{\perp}, T_{\parallel})$.

II.D (Continued)

2. Double Adiabatic Equations of State (CGL)

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{p_{s\perp}}{n_s B}\right) &= 0 \\ \frac{d}{dt}\left(\frac{p_{s\parallel} B^2}{n_s^3}\right) &= 0 \end{aligned}}$$

3. a. Since $p_{s\perp} = n_s k T_{\perp s}$, $\frac{p_{s\perp}}{n_s B} = \frac{n_s k T_\perp}{n_s B}$

b. But $v_{t\perp s}^2 = \frac{2kT_{\perp s}}{m_s}$, so $kT_{\perp s} = \frac{1}{2} m_s v_{t\perp s}^2 \Rightarrow \frac{p_{s\perp}}{n_s B} = \frac{m_s v_{t\perp s}^2}{2B} \simeq \mu$

c. Thus, we see the first of the equations is just a fluid form of <u>conservation of first adiabatic invariant</u>, the magnetic moment μ.

(Here $v_{\perp s}$ is replaced by $v_{t\perp s}$).

III. The Two Fluid Equations

1. Many phenomena in plasma physics can be described as two interpenetrating fluids, an ion fluid & an electron fluid.

2. Neglecting viscosity ($\underset{\sim}{\Pi} \to 0$) and taking Adiabatic Eq. of State ($\underset{\sim}{Q}_s \to 0$),

Continuity Equation: $\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underset{\sim}{U}_s) = 0$

Momentum Equation: $m_s n_s \left[\frac{\partial \underset{\sim}{U}_s}{\partial t} + \underset{\sim}{U}_s \cdot \nabla \underset{\sim}{U}_s\right] = -\nabla p_s + q_s n_s (\underset{\sim}{E} + \underset{\sim}{U}_s \times \underset{\sim}{B}) + \underset{\sim}{F}_{ps}$

Adiabatic Equation of State: $\frac{\partial}{\partial t}\left(\frac{p_s}{n_s^\gamma}\right) + \underset{\sim}{U}_s \cdot \nabla \left(\frac{p_s}{n_s^\gamma}\right) = 0$

For each species $s = i, e$

Charge Density: $\rho = \sum_s n_s q_s$

Current Density: $\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{j}_s$

Maxwell's Equations:

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

$\boxed{\text{16 Equations for 16 unknowns:}}$ $n_i, n_e, \underset{\sim}{U}_i, \underset{\sim}{U}_e, p_i, p_e, \underset{\sim}{E}, \underset{\sim}{B}$

IV. Generalized Ohm's Law

A. Small Electron Mass Approximation

1. An approximation frequently used in plasma physics is $\boxed{m_e \ll m_i}$
2. For Hydrogen, $\frac{m_i}{m_e} = 1836$, so this approximation is always well satisfied

B. Electron Momentum Equation

1. $$n_e m_e \left[\frac{\partial \underset{\sim}{U_e}}{\partial t} + \underset{\sim}{U_e} \cdot \nabla \underset{\sim}{U_e}\right] = -\nabla p_e - e n_e \left[\underset{\sim}{E} + \underset{\sim}{U_e} \times \underset{\sim}{B}\right] + \underset{\approx}{F_{De}}$$

2. a. Because $m_i \gg m_e$, the ion dominates the flow velocity

$$\underset{\sim}{U} = \frac{m_i \underset{\sim}{U_i} + m_e \underset{\sim}{U_e}}{m_i + m_e} \simeq \underset{\sim}{U_i}$$

b. But, the electrons are much more mobile, so $\underset{\sim}{U_i} - \underset{\sim}{U_e}$ gives rise to a current.

c. So, we can ~~neglect~~ let the ion velocity $\underset{\sim}{U_i} \simeq \underset{\sim}{U}$

$$\underset{\sim}{j} = e n_i \underset{\sim}{U_i} - e n_e \underset{\sim}{U_e} \underset{\uparrow}{=} e n_e (\underset{\sim}{U_i} - \underset{\sim}{U_e}) \simeq + e n_e (\underset{\sim}{U} - \underset{\sim}{U_e})$$

assume quasineutral plasma $n_i = n_e$

$$\Rightarrow \underset{\sim}{U_e} = \underset{\sim}{U} - \frac{\underset{\sim}{j}}{e n_e}$$

3. Because the electron mass is small, we can neglect the inertial terms on LHS as small, leaving

a. $$0 = -\nabla p_e - e n_e \left[\underset{\sim}{E} + \underset{\sim}{U_e} \times \underset{\sim}{B}\right] + \underset{\sim}{F_{De}}$$

b. Substituting for $\underset{\sim}{U_e}$,

$$-\frac{\nabla p_e}{e n_e} + \frac{\underset{\sim}{F_{De}}}{e n_e} = \underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} - \frac{\underset{\sim}{j} \times \underset{\sim}{B}}{e n_e}$$

c. $$\underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} = -\frac{\nabla p_e}{e n_e} + \frac{\underset{\sim}{j} \times \underset{\sim}{B}}{e n_e} + \frac{\underset{\sim}{F_{De}}}{e n_e}$$

Lecture #14 (Continued)

IV. B. (Continued)

4. But from Lecture #11, we know the collisional drag term

a. $\underset{\sim}{F}_{De} = m_e n_e \nu_{ei}(\underset{\sim}{U}_i - \underset{\sim}{U}_e) = \dfrac{m_e \nu_{ei}}{e}\Big[\underbrace{e n_e(\underset{\sim}{U}_i - \underset{\sim}{U}_e)}_{=\underset{\sim}{j}}\Big] = \dfrac{m_e \nu_{ei}}{e}\underset{\sim}{j}$

b. Thus $\dfrac{\underset{\sim}{F}_{De}}{e n_e} = \dfrac{m_e \nu_{ei}}{e^2 n_e}\underset{\sim}{j} = \eta \underset{\sim}{j}$ where $\boxed{\eta = \dfrac{m_e \nu_{ei}}{e^2 n_e}}$ Resistivity

5. Generalized Ohm's Law

$$\boxed{\underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} = \underbrace{\eta \underset{\sim}{j}}_{\text{Resistivity}} + \underbrace{\frac{\underset{\sim}{j} \times \underset{\sim}{B}}{e n_e} - \frac{\nabla p_e}{e n_e}}_{\text{Hall Terms}}}$$