Lecture 16: The Frozen-in Flux Theorem

I. Review: Basic Concepts in MHD

A. MHD Equations

Continuity Eq. \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \)

Momentum Eq. \( \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla (\rho + \frac{B^2}{2\mu_0}) + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{\mu_0} \)

Induction Eq. \( \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{\sigma}{\mu_0} \nabla^2 \mathbf{B} \)

Energy Eq. (Adiabatic) \( \frac{d}{dt} \left( \frac{\rho \mathbf{u}}{\gamma} \right) = 0 \)

1. Resistive MHD when \( \eta \neq 0 \)
2. Ideal MHD when \( \eta = 0 \)

B. MHD Approximation

1. Strong Collisions: \( \lambda_m \ll L, \gamma \gg (\frac{m_i}{m_e})^{\frac{1}{2}} \frac{1}{\sqrt{\eta}} \)
2. Non-relativistic: \( V_o \ll c \)
3. Magnetized: \( n_i \ll L \)

C. Properties of MHD

1. Quasineutrality: \( \frac{\rho}{e} n_i s_\perp \approx 0 \)
2. Ohm's Law: \( E + \mathbf{u} \times \mathbf{B} = \eta J \)

a. Typically, the conductivity of plasmas is very high, so \( \eta \) is small.

b. Usually, \( |\mathbf{E}| \ll |\mathbf{U} \times \mathbf{B}| \), so \( E \approx -\mathbf{U} \times \mathbf{B} \)

Thus, the fluid velocity \( \mathbf{U} \) corresponds to the \( \mathbf{E} \times \mathbf{B} \) velocity.
II. Dimensionless Numbers in Fluid Dynamics

A. General

1. In fluid dynamics, a common way to characterize the behavior of fluids in different physical systems is to calculate dimensionless numbers characteristic of the flow.

2. These dimensionless numbers typically characterize the magnitude of the ratio of two terms in the dynamical equations.

B. Example: Reynolds Number in Hydrodynamics, Re

1. Navier-Stokes Equation: \( \rho \frac{DU}{Dt} + \nabla \cdot \mathbf{U} = -\nabla p + \rho \nu \nabla^2 \mathbf{U} \)

where \( \rho \) is the density, \( \mathbf{U} \) is the velocity, \( \nabla \) is the gradient operator, and \( \nu \) is the kinematic viscosity.

2. We can divide by the density to yield:

\[ \frac{DU}{Dt} + \mathbf{U} \cdot \nabla \mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{U} \]

where we define

\[ \text{Kinematic Viscosity} \quad \nu = \frac{\mu}{\rho} \]

3. The Reynolds Number is defined as ratio of convection to diffusion term.

\[ \text{Re} = \frac{\nu \nabla U}{\nabla^2 U} \sim \frac{\nu}{\nu \nabla U / \nabla^2 U} \sim \frac{L V_0}{\nu} \]

b. Thus

\[ \text{Re} = \frac{L V_0}{\nu} \]

4. Low Re Flows (Re < (few hundred)) \( \Rightarrow \) laminar
High Re Flows (Re \( \geq \) 10^3) \( \Rightarrow \) turbulent.

C. Magnetic Reynolds Number: \( \text{Re}_m \)

1. Induction Equation: \( \frac{\partial B}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \frac{\mathbf{J}}{\mu_0} \nabla^2 \mathbf{B} \)

- Convection
- Diffusion
II. (Continued)

2. The Magnetic Reynolds Number is the ratio of convection to diffusion term in the induction equation,

\[ \text{Re}_m = \frac{\frac{\partial B}{\partial t} + \mathbf{v} \times \mathbf{B}}{\frac{\partial B}{\partial t} / \text{mol}^2} \sim \frac{\mu_0 L V_0}{\eta} \]

Thus \[ \text{Re}_m = \frac{\mu_0 L V_0}{\eta} \]

3. a. In the limit \( \text{Re}_m \gg 1 \), the convection term dominates and diffusion can be ignored \( \Rightarrow \) Ideal MHD.

b. In the limit \( \text{Re}_m \ll 1 \), the diffusion term dominates.

III. The Frozen-in Flux Theorem:

Most plasmas satisfy the condition \( \text{Re}_m \gg 1 \), giving

\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \]

Ideal MHD Induction Eq.

In this limit, a powerful theorem can be proven.

A. The Frozen-in Flux Theorem: The Magnetic Flux through a surface moving with the plasma (or fluid) velocity \( \mathbf{v}(x,t) \) remains constant.

**Proof:**

1. First, we define Magnetic Flux \[ \Phi_B = \int_S \mathbf{B} \cdot d\mathbf{A} \]

2. A Fact: The flux through any closed surface is zero.

\[ \int_S \mathbf{B} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{B} \, d^3x = 0 \]

Divergence Theorem, 
NRL 2.5 (25)
Lecture #16 (Continued)

III. A.2. (Continued)

b. Fact: The flux through any surface spanning a curve C is the same,

\[
\oint_{S_1} \mathbf{B} \cdot d\mathbf{A} = \oint_{S_2} \mathbf{B} \cdot d\mathbf{A}
\]

when \(d\mathbf{A}\) is taken in the same "sense."

3. a. Consider a surface \(S\) spanning curve \(C\) at time \(t\).

b. This moves to surface \(S'\) spanning curve \(C'\) at time \(t' = t + \Delta t\)

c. NOTE \(S' = S + \text{ribbon connecting } C \text{ to } C'\)

d. The area \(d\mathbf{A} = \Delta t \times \text{d}k\)

NOTE:

\[
\text{area } A = \alpha b \sin \alpha
\]

but \([a \times b] = \alpha b \sin \alpha\),

so \(A = \alpha b\)

4. Now, calculate \(\Phi_B\) at times \(t\) & \(t' = t + \Delta t\)

a. \(\Phi_B(t) = \oint_S \mathbf{B} \cdot d\mathbf{A}\)

b. \(\Phi_B(t + \Delta t) = \oint_{S'} \mathbf{B} \cdot d\mathbf{A} = \oint_S \mathbf{B}(t + \Delta t) \cdot d\mathbf{A} + \oint_{\text{ribbon}} \mathbf{B}(t + \Delta t) \cdot d\mathbf{A}\)

i) Expand \(\mathbf{B}(t + \Delta t) = \mathbf{B}(t) + \Delta t \frac{\partial \mathbf{B}}{\partial t} + \cdots \)

ii) For the ribbon \(\oint_{\text{ribbon}} \mathbf{B} \cdot d\mathbf{A} = \oint_{\mathbf{C}} \mathbf{B} \cdot (\mathbf{U} \times \text{d}k) = \Delta t \oint_{\mathbf{C}} (\mathbf{B} \times \mathbf{U}) \cdot \text{d}\mathbf{k}\)

\[
= \Delta t \oint_{\mathbf{C}} \nabla \times (k \times \mathbf{B}) \cdot d\mathbf{A}
\]

c. Dropping higher order terms,

\[
\Phi_B(t + \Delta t) = \oint_S \mathbf{B}(t) \cdot d\mathbf{A} + \Delta t \oint_{\text{ribbon}} \mathbf{B} \cdot d\mathbf{A} - \Delta t \oint_{\mathbf{C}} \nabla \times (k \times \mathbf{B}) \cdot d\mathbf{A}
\]
3. Theorem II: The magnetic field lines are frozen to the plasma/flux tube.

1. Consider a tube of plasma surrounding a magnetic field line at time $t$.

   No flux through sides of tube, Surface $S$

2. Let the tube move with fluid at velocity $U(x, y, t)$.

3. By Frozen-In Flux Theorem, at time $t'$, the flux through $S'$ must still be zero. This is true for all parts of the tube surfaces $S'$.

4. The flux through the ends must still be $\Phi$.

5. The field line still goes down the flux tube.

6. If we shrink the tube to infinitesimal size, flux tube encompasses a single field line.

Thus, the magnetic field lines are frozen to the fluid flow.
C. Clebsch Coordinates

Often representing the magnetic field in terms of scalar "potentials" is useful.

1. Example: Vacuum Scalar Potential for $\mathbf{B}$.
   a. In vacuum, $\mathbf{v} = 0$, so $\nabla \times \mathbf{B} = 0$ \implies \boxed{\mathbf{B} = -\nabla \phi}$
   b. The constraint $\nabla \times \mathbf{B} = 0$ then gives $\nabla^2 \phi = 0$ \quad \text{[Laplace's Equation]}

2. Clebsch Coordinates: (also called Flux Coordinates)
   a. Consider a reference surface that is everywhere perpendicular to the magnetic field.

   $\mathbf{B}$ is perpendicular to surface $S$

   b. Construct a coordinate system on $S$. For example, label each point on $S$ with $\psi$ and $\phi$.

   $\psi = \text{const}$
   $\phi = \text{const}$

   c. Continue $\psi$ and $\phi$ off the surface $S$ by making them constant along $\mathbf{B}$ field lines

   Thus and \boxed{$\mathbf{B} \cdot \nabla \psi = 0$ \quad $\mathbf{B} \cdot \nabla \phi = 0$}
4. Thus, we get: \[
\frac{\partial}{\partial \phi} \left( \nabla \times \nabla \phi \right) \cdot \nabla \phi = 0
\]
    \[
    = 0 \text{ if } \phi \neq \phi_0
\]
    \[
    \frac{2 \alpha}{\partial \phi} = 0
\]

5. Thus, \( \alpha = \alpha(\psi, \phi) \) [a function only of \( \psi \) and \( \phi \)].

6. To finish up derivation of Clebsh coordinates,

1. Let \( \phi = B \).
2. Define \( \alpha = \int^\psi_0 \alpha(\psi, B) d\psi \).

   Thus, \( \nabla \alpha = \int^\psi_0 \left( \frac{\partial \alpha}{\partial \psi} + \frac{\partial \alpha}{\partial B} \frac{d\psi}{dB} \right) d\psi = \alpha \nabla \psi + \alpha \frac{d\psi}{dB} \frac{d\psi}{dB} [\text{by NRL pg. 15}]
\]

   Thus, \( B = \nabla \alpha \times \nabla B = \left( \nabla \alpha + \alpha \nabla B \frac{d\psi}{dB} \right) \times \nabla B = \alpha \nabla \psi \times \nabla B \)

9. General Clebsh Representation: \[ B = \nabla \alpha \times \nabla B \]

1. Field lines are solutions of \( \alpha(\psi) = \text{constant}, \ B(\psi) = \text{constant} \).
2. NOTE: \( \nabla \times \nabla B = \nabla \times (\nabla \alpha \times \nabla B + \alpha \nabla \alpha \times \nabla B) = \nabla \alpha \times \nabla B = B \)
III. Applications of the Frozen-in Flux Theorem

A. Field Amplification

a. Incompressible Flow ⇒ Volume is the same.
b. Flux ratio is the same: \( \Phi_1 = B_1 A \quad \Phi_2 = B_2 \left( \frac{A_2}{A} \right) \)

⇒ \( B_2 = \frac{B_1 A}{A_2} = 2B_1 \) Amplified Field.

B. Zeldovich's Rope Dynamo

1. Circular Flux Tube

Stretch \( \Rightarrow \) Double larger ➔ Double \( B \)

Twist \( \Rightarrow \) Fold ➔ Tangle

2. Nearly the same as before, but \( B \) has doubled and there is a small tangle.