1. Review

A. Calculation of Linear Dispersion Relations

1. Linearize system of equations (e.g., assume $P = P_0 + \epsilon P_1 \text{ when } \epsilon \ll 1$)
2. Fourier Analysis: Find plane wave solutions $\psi \sim e^{i(k \cdot x - \omega t)}$
3. Write system of equations as matrix equation
   $\Rightarrow$ Solve determinant $= 0$ to yield $\omega = \omega(k)$.
4. We leave off with the following system

   1. $\omega p_1 = \rho_0 (k \cdot U_1)$
   2. $\omega U_1 = k \left( \frac{P_1}{\rho_0} + \frac{B_0 \cdot B_1}{M_0 \rho_0} \right) - \frac{(B_0 \cdot k) B_1}{M_0 \rho_0} \quad \text{(Note missing term in previous places)}$
   3. $\omega B_1 = B_0 (k \cdot U_1) - (B_0 \cdot k) U_1$
   4. $\omega p_1 = \gamma \rho_0 (k \cdot U_1)$

II. The MHD Dispersion Relations (Continued)

A. Let's simplify these equations

1. Eliminate $p_1$ from 2. using 4.

   2. $\omega^2 U_1 = k \left( \frac{\gamma \rho_0 (k \cdot U_1)}{\rho_0} + \frac{B_0 \cdot [k \cdot (B_1 \cdot B_1) / \rho_0 \rho_0]}{M_0 \rho_0} \right) - \frac{(B_0 \cdot k) [B_0 \cdot B_1]}{M_0 \rho_0}$

2. Now, use 3. to substitute for $\omega B_1$ in 2.

   3. $\omega^2 U_1 = k \left( \frac{\gamma \rho_0 (k \cdot U_1)}{\rho_0} + \frac{B_0 \cdot B_0 (k \cdot U_1) / \rho_0 \rho_0}{M_0 \rho_0} \right) - \frac{(B_0 \cdot k) [B_0 \cdot B_1]}{M_0 \rho_0} - \frac{(B_0 \cdot k) [B_0 \cdot B_1]}{M_0 \rho_0}$

3. Use $B_0 = B_0 \hat{k}$ to simplify further:

   $\omega^2 U_1 = k (k \cdot U_1) \left[ \frac{\gamma \rho_0}{\rho_0} + \frac{B_0^2}{M_0 \rho_0} \right] - k (B_0 \cdot U_1) (\hat{k} \cdot B_0) / M_0 \rho_0 - \frac{B_0^2 (B_0 \cdot k) [k \cdot U_1]}{M_0 \rho_0} + \frac{(B_0 \cdot k)^2 U_1}{M_0 \rho_0}$
Lecture 4B (Continued)

II. A. (Continued)

4. Define: DEF: Sound Speed \( c_s^2 = \frac{\varphi_0}{\rho_0} \)

Allen Speed \( V_{\lambda}^2 = \frac{B_0^2}{\mu_0 \rho_0} \)

\[ c_s^2 = (c_s^2 + V_{\lambda}^2) \hat{\mathbf{k}} \cdot (k \cdot \mathbf{U}) = V_k^2 (\hat{\mathbf{b}} \cdot \mathbf{U}) (\hat{\mathbf{b}} \cdot \mathbf{K}) k - V_k^2 (\hat{\mathbf{b}} \cdot \mathbf{K}) (k \cdot \mathbf{U}) b + V_k^2 (\hat{\mathbf{b}} \cdot \mathbf{K}) (k \cdot \mathbf{U}) b \]

5. Thus, we have reached a single (vector) equation for \( \mathbf{U} \).

b. Note: Once we have solved for \( \mathbf{U} \), \( \mathbf{B} \) is determined by \( \mathbf{U} \) using equation 1.

c. This vector equation represents 3 component equations. Thus, we can simplify to a matrix form:

\[
\begin{pmatrix}
3 \times 3 \text{ matrix}.
\end{pmatrix}
\begin{pmatrix}
U_x \\
U_y \\
U_z
\end{pmatrix} = 0
\]

But, first, we'll explore the solutions in simplified limits.

B. MHD Waves for \( k = k_{\parallel} \hat{\mathbf{b}} \) (Parallel Wave Vector \( k \parallel B_0 \))

1. In this case, we can simplify:

\[
(\hat{\mathbf{b}} \cdot \mathbf{U}) = k_{\parallel} U_z
\]

\[
\hat{\mathbf{b}} \cdot \mathbf{U} = U_z
\]

\[
\hat{\mathbf{b}} \cdot \mathbf{k} = k_{\parallel}
\]

where we take \( \mathbf{b} = \frac{\mathbf{b}}{\rho_0} \) and \( \mathbf{U} = U_\parallel \hat{\mathbf{b}} + U_y \hat{\mathbf{y}} + U_z \hat{\mathbf{z}} \).

2. Thus,

\[
c_s^2 U_{\parallel} = (c_s^2 + V_{\lambda}^2) k_{\parallel} U_z \hat{\mathbf{b}} - V_k^2 k_{\parallel} U_z \hat{\mathbf{b}} - V_k^2 k_{\parallel} U_z \hat{\mathbf{b}} + k_{\parallel} V_k^2 U_z \hat{\mathbf{b}} + k_{\parallel} V_k^2 U_z \hat{\mathbf{b}}
\]
3. Splitting into components and putting into matrix form:

\[
\begin{pmatrix}
\omega^2 - k_{||}^2 v_A^2 & 0 & 0 \\
0 & \omega^2 - k_{||}^2 v_A^2 & 0 \\
0 & 0 & \omega^2 - k_{||}^2 c_s^2
\end{pmatrix}
\begin{pmatrix}
U_x \\
U_y \\
U_z
\end{pmatrix}
= 0
\]

4. The determinant \( D = 0 \) is the dispersion relation:

\[
(\omega^2 - k_{||}^2 v_A^2)^2 (\omega^2 - k_{||}^2 c_s^2) = 0
\]

5. There are six solutions to this equation:

a. \( \omega = \pm k_{||} v_A \), \( k > 2 \), one for \( U_x \neq 0, U_y = 0 \), another for \( U_x = 0, U_y \neq 0 \).

b. \( \omega = \pm k_{||} c_s \)

6. Parallel Motions: \( \text{Sand Waves} \)

a. If we have \( U_z \neq 0 \), then \( \omega = \pm k_{||} c_s \)

\[
\begin{align*}
\text{Sand waves!} & \\
\text{\( u_x \to U_x \to U_z \to u_z \)} & \rightarrow B_0 \text{, These are the usual Sand waves moving along} \ B_0 \text{.} \\
\text{\( u_x \to U_x \to U_z \to u_z \)} & \rightarrow \text{at Sound Speed} \ c_s = \sqrt{\frac{\rho_0}{\rho}}
\end{align*}
\]

b. \( i(k_{||} z - \omega t) = i k_{||} (z \pm c_s t) \)

c. Since \( U_z = U_{z0} e^{ki_{\parallel}x} = U_{z0} e^{i k_{||}(z \pm c_s t)} \)

d. \( B \) is unperturbed by motion along \( B_0 \).

e. In this limit of \( k = k_{||} \hat{\alpha} \), the relevant equations are

- Component of Momentum eqs: \( \rho \frac{du_z}{dt} = \rho \frac{dU_z}{dx} - \frac{2}{5} \pi \)
- Pressure equation: \( \frac{dP}{dx} = -\rho \frac{dU_z}{dx} \)

Parallel motion \( U_z \) leads to compression \( \frac{P_{ii}}{\gamma} \) to \( \frac{P_{ii}}{\gamma} \) by \( \pi \) in \( P_{ii} \) acts as restoring force \( \Rightarrow \) the usual Sound wave!
Lecture 4/8 (Continued)

II. B. (Continued)

Alfvén Waves

7. Perpendicular Motions:
   a. For \( U_y \neq 0 \), we must have \( c = \pm k_x V_A \).

b. Alfvén waves are like waves on a string, propagating at

Relevant equations:

\( \text{\( x \)-component of Momennun Eq:} \quad p_0 \frac{\partial U_x}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial B_x}{\partial z} \)  
Magnetic Tension term

\( \text{\( x \)-component of Induction Eq:} \quad \frac{\partial B_x}{\partial t} = B_0 \frac{\partial U_x}{\partial z} \)

Motion \( U_{xx} \) is perpendicular to \( B_0 \), causing it to bend
Magnetic tension acts as restoring force

d. Because \( k \cdot U_i = 0 \), this motion is incompressible.

e. We could also have taken \( U_y \neq 0 \) with \( U_x = 0 \), and results are analogous.

Two polarizations of Alfvén Wave
in direction perpendicular to \( B_0 \).

C. MHD Waves for \( k = k_x = k_1 \hat{\mathbf{x}} \) (Perpendicular Wavevector \( k \perp B_0 \))

1. In this case
   \[
   (k \cdot U_i) = k_1 U_x \\
   (\hat{\mathbf{\mu}} \cdot U_i) = U_z \\
   (\hat{\mathbf{\mu}} \cdot k) = 0
   \]

2. Thus \( c^2 U_i = (c^2 + \omega^2) k_1^2 U_x \hat{\mathbf{x}} + 0 \)
3. Splitting into Component Form:
\[
\begin{pmatrix}
\omega^2 - k_1^2 (c_s^2 + V_h^2) & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega^2
\end{pmatrix}
\begin{pmatrix}
U_x \\
U_y \\
U_z
\end{pmatrix} = 0
\]

4. The determinant \( \Delta = 0 \) gives
\[
\omega^4 - [\omega^2 - k_1^2 (c_s^2 + V_h^2)] = 0
\]

5. Again we have six solutions:
   a. For solutions with \( \omega = 0 \)
      \( \text{for } U_y \neq 0 \text{ or } U_z \neq 0 \)
   b. Two solutions with \( \omega = \pm k_1 (c_s^2 + V_h^2)^{1/2} \)
      \( \text{if } U_x \neq 0 \)

6. Zero Frequency Solutions:
   a. For \( U_z \neq 0 \)
      \( \omega \rightarrow \omega \rightarrow \omega \rightarrow \omega \rightarrow \omega \rightarrow \omega \)
      i. These are like sound waves
         (motion along field)
      ii. But \( k_1 U_z = 0 \), so no compression, and thus no restraining force.
   b. For \( U_y \neq 0 \), motion is in \( \hat{y} = B_0 \times \hat{k}_1 \) direction.
      i) Magnetic field lines may slide past one another. Again, no restraining force, so \( \omega = 0 \).
      ii) This is called an interchange mode: It moves straight magnetic field lines with bending them.

7. Magneto-Acoustic (or Fose) Wave \( U_x \neq 0 \) \( \Rightarrow \omega = \pm k_1 (c_s^2 + V_h^2)^{1/2} \)

Compression

\[
\begin{pmatrix}
U_x \\
U_y \\
U_z
\end{pmatrix} \rightarrow B_0 \]
a. Motions are similar to compressional sound waves, but include a contribution from the magnetic pressure as well, propagating at speed \( (c_s^2 + V_h^2)^{1/2} \).
b. Relevant Equations:

- Component of Momentum Eq. \[ \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \mathbf{b} \times \mathbf{B} \]
- Component of Induction Eq. \[ \frac{\partial \mathbf{B}}{\partial t} = -\mathbf{J} \times \mathbf{B} \]

Pressure Equation \[ \frac{\partial P}{\partial t} = -\rho \frac{\partial \mathbf{U}}{\partial x} \]

c. \( c_U = k_x U_x \), so these waves are compressional.

d. Perpendicular motion \( U_x \) compresses both plasma and magnetic field. Restoring force includes both thermal pressure \( P \) and magnetic pressure due to \( B_z \).

e. **Note:** A fluctuation with only \( B_z \neq 0 \) has \( \mathbf{B} = (B_0 + B_z) \). Magnetic field does not change direction but does increase magnitude.

D. The General Case of MHD Dispersion Relation

1. We can solve the MHD Dispersion Relation for any wavevector \( \mathbf{k} \).
   a. Without loss of generality, we take \( \mathbf{k} = k_x \mathbf{e}_x + k_z \mathbf{e}_z = k \sin \theta \mathbf{e}_x + k \cos \theta \mathbf{e}_z \)

where \( k = k_z \)

b. In general, then, \( U_c \mathbf{U} = k \sin \theta \mathbf{U}_x + k \cos \theta \mathbf{U}_z \)
   \[ k \mathbf{k} = k \cos \theta \]
   \[ \mathbf{U}_c \mathbf{U} = \mathbf{U}_z \]
2. After some algebra, the dispersion relation is found to be:

\[
(\omega^2 - k^2 \cos^2 \theta \nu_A^2) \left[ \omega^4 - \omega^2 k^2 (\nu_c^2 + \nu_A^2) + k^4 \cos^2 \theta \nu_c^2 \nu_A^2 \right] = 0
\]

General MHD Dispersion Relation

3. Six Solutions: Three Waves, each with $\Theta$ & $\Phi$.

4. Alfvén Waves: $c_s^2 = k^2 \cos^2 \theta \nu_A^2 \Rightarrow c_s^2 = k_{\parallel}^2 \nu_A^2$

Motion is in the $\hat{b} \times \hat{k}$ direction ($\Theta$ direction)

3D Picture:

Polarization:
1. Motion is out of the plane defined by $B_0$ and $k$.
2. Resisting force is only magnetic tension.
3. $k \cdot k' = 0 \Rightarrow$ Alfvén wave is incompressible.

Sometimes called the "Shear Alfvén Wave."
Lecture #18 (Continued)

II. Co (Continued)

5. Fast Waves

\[ \frac{c^2}{k^2} = \frac{1}{2}(c_s^2 + V_A^2) + \frac{1}{2} \sqrt{(c_s^2 + V_A^2)^2 - 4c_s^2 V_A^2 \cos^2 \theta} \]

\[ \text{phase speed } v_p = \frac{c^2}{k} \]

\[ \text{waves} \]

b. This wave is a mixture of Compressional wave and Transverse wave.
   - Resonating forces:
     1) Thermal and Magnetic pressure add together.
     2) Bonding of field lines - magnetic tension

Polarization a. Motion is in the plane of \( \mathbf{B}_0 \) and \( \mathbf{k} \)
Has both \( \hat{b} \) (\( = \hat{z} \)) and \( \hat{c} \) (\( \perp \mathbf{k} \)) components \( U_z \) & \( U_x \)

b. This wave is a mixture of Compressional wave and Transverse wave.
   - Resonating forces:
     1) Thermal and Magnetic pressure add together.
     2) Bonding of field lines - magnetic tension

d. For \( \theta = 0 \), \( c^2 = k^2 c_s^2 + \sqrt{k^2 c_s^2 + \frac{c_s^2}{k^2} V_A^2} \) Sand Wave
   \( k^2 V_A^2 \) \( V_A > c_s \) Alfvén Wave

2. For \( \theta = \frac{\pi}{2} \), \( c^2 = k^2 (c_s^2 + V_A^2) \) Magneto-acoustic wave
Slow Waves

\[ \frac{c^2}{k^2} = \frac{1}{2} (c_s^2 + v_A^2) - \frac{1}{2} \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2 \theta} \]

2D Picture

Polarization

- Motion is in the plane of \( B_0 \) and \( k \)
- Has both \( \hat{z} (= \hat{z}) \) and \( k_z (= \hat{z}) \) components \( U_z \) & \( U_x \)

b. Wave is a mixture of compressional and transverse motions

- Resisting Force: 1) Thermal pressure and Magnetic pressure oppose
  2) Magnetic tension due to bending of field lines

c. Resisting Force is weak because thermal and magnetic pressures subtract
  \( \Rightarrow \) Wave is slow

d. For \( \theta = 0 \),

\[ c^2 = \frac{k^2 c_s^2}{k^2 v_A^2} \quad \text{Sound Wave} \]

\[ c^2 < v_A^2 \]

\[ c^2 > v_A^2 \quad \text{Alfvén Wave} \]

e. For \( \theta \to 0 \),

\[ c^2 \to 0 \]

Magnetic and thermal pressures subtract completely.