Lecture 19: More About MHD Waves

I. Review

All last time, we linearized the MHD equations, assumed plane wave (Fourier) solutions, and solved to obtain the MHD dispersion relation:

\[
(a^2 - k^2 \sin^2 \theta V^2)(a^4 - a^2 k^2 (c_s^2 + V_h^2) + k^4 \sin^2 \theta c_s^2 V_h^2) = 0
\]

where

\[
\mathbf{B}_0 \cdot \mathbf{k} = B_0 \cos \theta
\]

B. Three Wave Modes:

1. Alfvén Waves:
   a. \( c_0^2 = \frac{k^2 V_h^2}{a^2} \)
   b. Motion out of the plane defined by \( \mathbf{B}_0 \) and \( \mathbf{k} \)
   c. Incompressible
   d. Resolving Force: Magnetic Tension alone

2. Fast Waves:
   a. \( \frac{c_0^2}{c^2} = \frac{1}{2} (c_s^2 + V_h^2) - \frac{1}{2} \sqrt{(c_s^2 + V_h^2)^2 - 4 c_s^2 V_h^2 \cos^2 \theta} \)
   b. Motion in the plane of \( \mathbf{B}_0 \) and \( \mathbf{k} \)
   c. Compressible (usually)
   d. Resolving Force: i) Thermal and Magnetic Pressure Add! ii) Magnetic Tension

3. Slow Waves:
   a. \( \frac{c_0^2}{c^2} = \frac{1}{2} (c_s^2 + V_h^2) + \frac{1}{2} \sqrt{(c_s^2 + V_h^2)^2 - 4 c_s^2 V_h^2 \cos^2 \theta} \)
   b. Motion in the plane of \( \mathbf{B}_0 \) and \( \mathbf{k} \)
   c. Compressible
   d. Resolving Force: i) Thermal and Magnetic Pressure Subtract! ii) Magnetic Tension
II. Polar Plot of MHD Wave Phase Speeds:
A. Dimensionless Version of MHD Dispersion Relation
1. Take \( k_x = k \cos \theta \) and \( k_z = k \sin \theta \).
2. Normalize by dividing by \( a_c^2 \):
\[
\left( \frac{\omega^2}{a_c^2} - k_x^2 \frac{V_A^2}{a_c^2} \right) \left[ \frac{\omega^4}{a_c^4} - \frac{\omega^2}{a_c^2} \left( k_x^2 + k_z^2 \right) V_A^2 \left( 1 + \frac{C_s^2}{V_A^2} \right) + k_x^2 (k_x^2 + k_z^2) \frac{V_A^4}{a_c^4} \right] = 0
\]
3. **NOTE:** Let \( \tilde{a} = \frac{a_c}{a_{i1}} \)

   a. \( \frac{V_A^2}{a_{i1}^2} = \frac{B_0^2}{\mu_0 \rho_i} = \frac{1}{\mu_0 \rho_i} \left( \frac{m_i}{e^2} \right) \frac{C_s^2}{a_{i1}^2} \Rightarrow \text{This is the ion inertial length.} \)

   **DEFINITION:** \( d_i = \frac{a_{i1}}{\omega p_i} = \frac{V_A}{C_s} \)

   b. \( C_s^2 = \frac{\gamma}{\rho_i} \frac{2 \mu_0 \rho_i}{B_0^2} = \frac{\gamma}{2} \beta \Rightarrow \text{Plasma } \beta. \)

   c. \( \beta = \frac{Z_n p_n}{B_0^2} \)

4. Thus,
\[
\left( \tilde{\omega}^2 - k_x^2 \frac{d_i^2}{a_{i1}^2} \right) \left[ \tilde{\omega}^4 - \tilde{\omega}^2 \left( k_x^2 + k_z^2 \right) \left( 1 + \frac{\gamma}{2} \beta \right) + k_z^2 \left( k_x^2 + k_z^2 \right) \right] = 0
\]

5. There are only three parameters (dimensionless) which \( \tilde{\omega} \) depends on:

   \[ \tilde{\omega} = \tilde{\omega}_{k \phi_1} \left( k_x d_i, k_z d_i, \beta_i \right) \]

   a. Two define the parallel & perpendicular components of the wavenumber (this is characteristic of most dispersion relations)

   b. Only one other dimensionless parameter: \( \beta_i \)

6. **NOTE:** \( a_i = \frac{n_i}{\sqrt{\beta_i}} \) when \( \beta_i = \frac{Z_n p_i}{B_0^2} = \frac{\beta}{2} \) for \( T_i = T_e \) (true for MHD)

   b. Thus, we can write \( \tilde{\omega} = \tilde{\omega}_{k \phi_1} \left( k_x d_i, k_z d_i, \beta_i \right) \)
7. Validity of MHD Approximation:
   a. Remember \( n_i \ll L \), so if \( L \sim \frac{1}{k} \), this means \( kn_i \ll 1 \).
   b. Also, \( v_0 = \frac{L}{\tau} \Rightarrow n_i \ll L = \tau v_0 \)
      i. For \( v_0 \sim v_{i1} \) and using \( n_i = \frac{v_{i1}}{c_{i1}} \), we get \( \frac{v_{i1}}{c_{i1}} \ll \tau v_{i1} \).
      ii. Take \( c_{\phi} \sim \frac{1}{\tau} \), giving us \( c_{\phi} \ll c_{s1} \).
   c. Thus \( \tilde{c}\phi = \tilde{c}_{\phi,0} (k_{n1} n_i, k_{\nu1} n_i, \beta_i) \) is valid when \( \tilde{c}\phi \ll 1 \).

B. Limits of \( \frac{c_{\phi}}{v_{A}} \) at \( \theta = 0 \):
1. Phase velocity \( v_p = \frac{c_{\phi}}{v_{A}} \) for waves at \( \theta = 0 \)

\[
\begin{align*}
\text{Fast} & \quad \frac{c_{\phi}^2}{v_{A}^2} = c_s^2 \\
\text{Alfvén} & \quad \frac{c_{\phi}^2}{v_{A}^2} = v_{A}^2 \\
\text{Slow} & \quad \frac{c_{\phi}^2}{v_{A}^2} = \frac{c_s^2 v_{A}^2}{v_{A}^2}
\end{align*}
\]

C. Polar Plots of \( \frac{c_{\phi}^2}{v_{A}^2} \):
\[\theta = 0^\circ \]
\[\theta = \frac{\pi}{2} \]
\[\theta = \pi \]

\[\frac{\gamma}{2} \beta > 1 \quad \text{HIGH BETA} \]
\[\frac{\gamma}{2} \beta < 1 \quad \text{LOW BETA} \]
III. Conservation of Energy in Ideal MHD:

A. The MHD Equations can be manipulated to give a law for the Conservation of Energy:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} U^2 + \frac{\rho}{\delta-1} + \frac{B^2}{2 \mu_0} \right) + \nabla \cdot \left( \frac{1}{2} \mu_0 U^2 + \frac{\rho}{\delta-1} \nabla U + \frac{1}{\mu_0} F \times B \right) = 0
\]

Kinetic Energy | Internal (Thermal) Energy | Magnetic Energy | Flux of Kinetic Energy | Enthalpy Flux | Poynting Flux

2. Integrating over all space, the volume integral of 2nd term can be converted to a surface integral by divergence theorem.

\[
\frac{dE}{dt} = 0
\]

with

\[
E = \frac{1}{2} \mu_0 U^2 + \frac{\rho}{\delta-1} + \frac{B^2}{2 \mu_0}
\]

Conserved Energy in Ideal MHD.

IV. The Entropy Model:

A. The MHD Equations give 8 equations for 8 unknowns: \( \rho, U, B, P \).

2. But we found only 6 solutions to the dispersion relation.

3. In fact, a more careful analysis give two roots with \( \omega = 0 \).

What do these roots correspond to?

B. Divergencelessness of \( B \):

1. Remember, we must always satisfy \( \nabla \cdot B = 0 \), so there is really an additional constraint, so we only have 7 unknowns, and this seven solutions.

C. The Entropy Model:

1. We define \( \text{DEF: Specific Entropy} \) \( S = \frac{C_p}{\rho} \) where \( C \) is some constant.
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II(Concluded)

2. Thus, the adiabatic Equation Stear is $\frac{\partial s}{\partial t} = 0$.
   $\Rightarrow$ Thus, entropy is conserved by these adiabatic fluctuations.

3. If we are considering fluctuations, $p = p_0 + p_1$
   $s = s_0 + s_1$, etc.

   a. The other $c=0$ mode is a zero frequency energy mode
   $s_1 \neq 0$, but $p_1 = 0$ (and so are $\gamma = 0$ & $\beta_1 = 0$).

4. Consider the ideal gas law: $pV = NkT$, or $p = nkT = \rho kT$
   a. We can have $\rho = 0$ if $\rho T_1 = \text{const.}$
   b. Thus, density & temperature can vary to give constant pressure, $p = \text{const}$.

   5. The existence of the (cold-regulated) Entropy mode should not be forgotten.

   6. There are 7 solutions to ideal MHD dispersion relation
      a. Six waves ($\pm$ Fast, $\pm$ Alfvén, $\pm$ Slow)
      b. One zero-frequency entropy mode

V. Eigenfunctions of the MHD Eigenmodes

A. How do we determine eigenfunctions $(p_0, u_1, b_1, p_1)$ for a given wave mode?

1. We need to go back to the simplified matrix equation for MHD.
2. Choose a value for one component.
3. Solve for all other quantities.
3. Examples: Eigenfunctions for \( k_{11} = k_1 = k_0 \) (\( \theta = 45^\circ \))

1. In this case, the vector equation for \( \mathbf{U} \) is

\[
\begin{pmatrix}
\omega^2 - k_0^2(c_0^2 + 2v_k^2) & 0 & -k_0^2c_s^2 \\
0 & \omega^2 - k_0^2v_k^2 & 0 \\
-k_0^2c_s^2 & 0 & \omega^2 - k_0^2c_s^2
\end{pmatrix}
\begin{pmatrix}
U_x \\
U_y \\
U_z
\end{pmatrix} = 0
\]

2. Noted: As clear above, \( U_y \) is decoupled from \( U_x \) and \( U_z \).
3. Let's find the fast wave eigenfunction for \( U_y = U_0 \) (Take \( U_y = 0 \))

\[
\begin{pmatrix}
\omega^2 - k_0^2(c_0^2 + 2v_k^2) & -k_0^2c_s^2 \\
-k_0^2c_s^2 & \omega^2 - k_0^2c_s^2
\end{pmatrix}
\begin{pmatrix}
U_x \\
U_z
\end{pmatrix} = 0
\]

b. I can use either equation to solve for \( U_z \) as a function of \( U_x \).

\[-k_0^2c_s^2 U_x + \omega^2 - k_0^2c_s^2 U_z = 0\]

\[\Rightarrow U_z = \frac{k_0^2c_s^2}{\omega^2 - k_0^2c_s^2} U_x\]

\[U_x = U_0\]

3. Here, for the fast slow wave,

\[c_0^2 = \frac{1}{2} k_0^2(c_0^2 + v_k^2) \pm \frac{k_0^2}{2} \sqrt{(c_0^2 + v_k^2)^2 - 2c_0^2v_k^2} = \pm \frac{k_0^2(c_0^2 + v_k^2)}{2}\]

Noted: The wave speed is \( c_0 \), which is fast or slow.

4. Find density perturbation: \( \rho_1 = \rho_0(k_0 U_1) = \rho_0(k_0 U_x + k_0 U_z) \)

\[\rho_1 = \rho_0 \frac{k_0}{c_0^2} (U_x + U_z) = \rho_0 \frac{k_0}{c_0^2} (U_0 + \frac{k_0^2c_s^2}{\omega^2 - k_0^2c_s^2} U_x) = \rho_0 \frac{k_0}{c_0^2} \frac{\omega^2 - k_0^2c_s^2 U_x}{c_0^2 - k_0^2c_s^2 U_0} \]
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4. Similarly:

\[ P_1 = P_0 \frac{c_k u_0}{c_l^2 - k_o c_s^2} U_0 \]

6. Magnetic field:

\[ \omega B_z = B_0 (k_o U_x + k_o U_z) - B_0 k_o U_x = B_0 k_o U_x \]

a. \( \omega B_x = -B_0 k_o U_x \)

b. \( \omega B_z = B_0 (k_o U_x + k_o U_z) - B_0 k_o U_x = B_0 k_o U_x \)

c. Thus:

\[
\begin{align*}
    B_x &= -B_0 \frac{k_o}{\omega} U_0 \\
    B_z &= B_0 \frac{k_o}{\omega} U_0
\end{align*}
\]

d. **Note:** \( V \cdot B \Rightarrow E \cdot B_1 = k_o (B_x + k_o B_z) = k_o (-B_0 \frac{k_o}{\omega} U_0 + B_0 k_o U_z) = 0 \)

7. Thus, for the case with \( k = k_o \hat{x} + k_o \hat{z} \), we get:

\[
\begin{align*}
    P_1 &= P_0 \frac{c_k u_0}{c_l^2 - k_o c_s^2} U_0 \\
    P_1 &= V_0 \frac{c_k c_s}{c_l^2 - k_o c_s^2} U_0 \\
    U_x &= U_0 \\
    U_y &= 0 \\
    U_z &= \frac{k_o c_s^2}{c_l^2 - k_o c_s^2} U_0 \\
    B_x &= -B_0 \frac{k_o}{\omega} U_0 \\
    B_z &= B_0 \frac{k_o}{\omega} U_0
\end{align*}
\]

8. Let's look at the total pressure term for fast/slow wave in the simple case, \( c_s^2 = V_A^2 \).
\[ \frac{1}{\rho_0} \sqrt{p + \frac{B^2}{2u_0}} \Rightarrow k \left( \frac{p_1}{\rho_0} + \frac{B_{e0}B_1}{\rho_0 \mu_0} \right) \]

6. Since \( k = k_0 \frac{e}{k_0} \), both components of pressure gradient are the same:

\[ k_0 \left[ \frac{c_{w_0}^2 \frac{c_{w_0}^2}{\omega_0^2 + k_0c_{s_0}^2}}{\rho_0} + \frac{B_{e0}B_0}{\rho_0 \mu_0} \right] \]

\[ = k_0 \left( \frac{c_{s_0}^2 \frac{c_{s_0}^2}{c_{s_0}^2 + k_0c_{s_0}^2}}{\omega_0} \right) \frac{k_0}{c_{w_0}} U_0 \]

\[ c_{w_0}^2 = k_0^2 c_{s_0}^2 \left[ 1 \pm \frac{k_0^2 c_{s_0}^2}{c_{s_0}^2} \right] \]

\[ \omega_0 = k_0^2 c_{s_0}^2 \left( \frac{c_{w_0}^2}{c_{w_0}^2 + k_0c_{s_0}^2} + 1 \right) U_0 \]

\[ c_{w_0}^2 - k_0^2 c_{s_0}^2 = k_0^2 c_{s_0}^2 \left( \frac{\pm k_0^2 c_{s_0}^2}{c_{s_0}^2} \right) = \pm k_0^2 c_{s_0}^2 \]

7. Thus, the pressure term is

\[ \frac{k_0^2 c_{s_0}^2}{\omega} \left( \pm \frac{k_0^2 c_{s_0}^2}{c_{s_0}^2} + 1 \right) U_0 = \sqrt{2} c_{w_0} \left( 1 \pm \frac{\sqrt{2}}{2} \right) U_0 \]

\[ \begin{cases} 2(\pm \sqrt{2}) c_{w_0} U_0 & \text{FAST} \\ 2 c_{w_0} U_0 & \text{SLOW} \end{cases} \]