Lecture #24 Cold vs. Warm Unmagnetized Plasma Waves

I. Longitudinal Modes in Warm vs Cold Unmagnetized Plasmas

A. Longitudinal Modes

1. Last time, we found that, in a cold plasma, longitudinal modes, or Plasma Oscillations, do not propagate.

2. If a plasma has a finite temperature — typically the case in laboratory, space, and astrophysical plasmas — longitudinal modes do propagate as Langmuir Waves or Ion acoustic Waves.

3. To demonstrate this, we will calculate the dispersion relation when the electron temperature is finite, \( T_e \neq 0 \).
   a. We will take \( T_i = 0 \) (or \( T_i < T_e \)), a cold ion approximation to keep the mathematics simple.
   b. In laboratory plasmas, it is common to have \( T_i < T_e \).

**Aside:** Why?

1) To ionize a laboratory plasma, often a hot cathode is used.

2) The hot cathode emits very fast electrons which stream along the magnetic field line, leading to ionization of the plasma (collisional ionization).

3) From Lecture #1, we learned about the timescale to achieve thermal equilibrium between species via collisions.
   a. Plasma electrons can easily gain energy from (hot electron) \(-\) (electron) collisions on a timescale \( T_{ee} \approx \frac{1}{n_e} \).
   b. But, for ions to reach thermal equilibrium with hot electrons requires a time \( T_{iz} \approx \frac{m_i}{n_e} T_{ee} \approx 1836 T_{ee} \gg T_{ee} \).
II. Dispersion Relation for Non-Unmagnetized Plasma with $T_i \ll T_e$

A. Equations:

1. We may not neglect the electron pressure term in the electron momentum equation, but we'll continue to neglect ion pressure.

- Continuity: \[ \frac{\partial n_i}{\partial t} + \nabla \cdot \mathbf{v}_i = -n_i \nabla \cdot \mathbf{U}_i \]
- Electron equation: \[ \frac{\partial n_e}{\partial t} + \mathbf{v}_e \nabla \cdot n_e = -n_e \nabla \cdot \mathbf{U}_e \]

- Momentum: \[ \mathbf{m}_i \frac{\partial \mathbf{U}_i}{\partial t} + \mathbf{U}_i \nabla \cdot \mathbf{m}_i = q_i n_i (\mathbf{E} + \mathbf{U}_i \times \mathbf{B}) \]
- Electron equation: \[ \mathbf{m}_e \frac{\partial \mathbf{U}_e}{\partial t} + \mathbf{U}_e \nabla \cdot \mathbf{m}_e = -\nabla \mathbf{p}_e + q_e n_e (\mathbf{E} + \mathbf{U}_e \times \mathbf{B}) \]

Maxwell's Equations:
- \[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]
- \[ \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \]
- \[ \nabla \cdot \mathbf{B} = 0 \]
- \[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

2. Closure: We need an additional equation to relate \( \mathbf{p}_e \) to the other variables to close this system.

3. Isothermal Equation of State:

   a. \[ \frac{d}{dt} \left( \frac{\mathbf{p}_e}{n_e} \right) = 0 \]

   b. This time we take \( \gamma = 1 \), corresponding to an isothermal eq. of state.

   c. \( \mathbf{p}_e = n_e C \)

   d. Because electrons move very rapidly in any direction in an unmagnetized plasma, for slow motions (low freq.) an isothermal eq. of state is often used.
II. (Continued)

B. Linearization and Fourier Transform of Equations

1. This has been done in detail in Lecture #22, so we'll only treat the new electron momentum equation.

2. Assume:
   \( N_e = N_{e0} + e \cdot n_{ei} \)
   \( U_e = eU_{ei} \)
   \( n_e = n_{e0} + e \cdot p_{ei} \)
   \( \mathbf{E} = eE_1 \)
   \( \mathbf{B} = eB_1 \)

3. Linearize:
   \( \mathbf{E} = n_{e0} \frac{\partial U_{ei}}{\partial t} + e^2 n_{e0} \frac{\partial \mathbf{E}_1}{\partial t} + e^2 U_{ei} \cdot \nabla \mathbf{V}_{ei} + e^2 n_{e0} \mathbf{V}_{ei} \cdot \nabla \mathbf{E}_1 \)

4. \( \nabla \cdot \mathbf{E} = \text{constant} \)

5. Fourier Transform:
   All plasma wave solutions \( \propto e^{-i(k \cdot \mathbf{x} - \omega t)} \)

   a. \( i n_{e0} \omega U_{ei} = -i \mathbf{K}_{pe} + \omega n_{e0} \mathbf{E}_1 \)

   b. \( \omega U_{ei} = \frac{\mathbf{K}_{pe}}{n_{e0}} + \frac{i \omega n_{e0}}{m_e} \mathbf{E}_1 \)
C. System of Equations:

1. \( \omega n_{i1} = n_{i0} k \cdot U_{i1} \)
2. \( \omega ne_1 = ne_0 k \cdot U_{e1} \)
3. \( \omega U_{i1} = \frac{\mu_0}{m_i} E_1 \)
4. \( \omega U_{e1} = \frac{k}{\omega e e_0} \frac{pe_1}{me e_0} + i \frac{q e}{me} E_1 \)
5. \( i k \cdot E_1 = \frac{1}{\epsilon_0} (q i n_{i1} + q e ne_1) \)
6. \( \omega B_1 = k \times E_1 \)
7. \( \omega E_1 = \frac{i}{\epsilon_0} (q i n_{i1} U_{i1} + q e ne_0 U_{e1}) - c^2 k \times B_1 \)
8. \( pe_1 = ne_0 kT_e \)

D. Solve in terms of \( E_1 \) only to yield maxwell's equations:

1. Using 1, 3, 6, and 8, we can solve 4 in terms of \( U_{e1} \) and \( E_1 \) only. The result is:

\[
\omega U_{e1} = \frac{iq e}{me} \left[ \frac{\epsilon_0^2}{\omega \rho_s^2} \left( 1 - \frac{\omega^2}{\omega_p^2} \right) \right] k (k \cdot E_1) + E_1
\]

where we have used the definition \( \epsilon_0^2 = \frac{k T_e}{m e} \)

and definitions for the plasma frequency \( \omega_p = \sqrt{\frac{n_i q_i^2}{e_m}} \)

2. Using 3, 6, and 9, we can solve 7 solely in terms of \( E_1 \), obtaining:

\[
\left[ a^2 - \omega_p^2 - c^2 k^2 \right] E_1 - \left[ \epsilon_0^2 \left( 1 - \frac{\omega^2}{\omega_p^2} \right) + c^2 \right] k (k \cdot E_1) = 0
\]

3. **NOTE:** \( E_1 = E_{i1} \hat{E}_i + E_{e1} \hat{E}_e + E_{s1} \hat{E}_s \)

and

\[
k (k \cdot E_1) = k^2 \frac{k}{k} (k \cdot E_1) = k^2 E_{s1} \hat{S}
\]

where \( \omega_p = \omega_{ip}^2 + \omega_{epe}^2 \)
II.D. (Continued)

4. The three equations are then:

\[
\begin{align*}
\frac{\partial^2 E_1}{\partial t^2} - (\omega^2 - \omega_p^2 - c^2 k^2) E_{T1} &= 0 \\
\frac{\partial^2 E_2}{\partial t^2} - (\omega^2 - \omega_p^2 - c^2 k^2) E_{T2} &= 0 \\
(\omega^2 - \omega_p^2 - c^2 k^2 - \frac{c^2 k^2}{\omega^2})(1 - \frac{\omega_p^2}{\omega^2}) + c^2 k^2) E_L &= 0
\end{align*}
\]

5. Thus, our Matrix equation is:

\[
\begin{pmatrix}
\omega^2 - \omega_p^2 - c^2 k^2 & 0 & 0 \\
0 & \omega^2 - \omega_p^2 - c^2 k^2 & 0 \\
0 & 0 & \omega^2 - \omega_p^2 - k^2 c^2 (1 - \frac{\omega_p^2}{\omega^2})
\end{pmatrix}
\begin{pmatrix}
E_T1 \\
E_T2 \\
E_L
\end{pmatrix} = 0
\]

a. **NOTE!** The only change from the Cold Plasma Equations is in the Longitudinal term.

b. The inclusion of electron pressure modifies the plasma oscillation.

b. The transverse Modified Light Waves are unaffected by the pressure gradient term. This is because \( E_L = 0 \) for these modes, so there are no density variations and thus no pressure fluctuations for the transverse mode.

6. Longitudinal Term:
\[
\begin{align*}
(\omega_L^2 - \omega_p^2 - k^2 c^2 (1 - \frac{\omega_p^2}{\omega^2}) &= 0
\end{align*}
\]

a. General Solution:
\[
\omega^2 = \frac{(\omega_p^2 + k^2 c^2)}{2} \left( 1 \pm \sqrt{1 - \frac{4 k^2 c^2 \omega_p^2}{(\omega_p^2 + k^2 c^2)^2}} \right)
\]

b. There are four separate roots, corresponding to two wave modes:

1. Langmuir Waves
2. Ion Acoustic Waves
c. In general, the solution is difficult to interpret.
   \[ \Rightarrow \text{Plasma physicists often take limits of the solution.} \]

Finding simplified solutions in limiting cases is an important skill for interpreting physical behavior.

d. For example:
\[ \omega^2 - \frac{\omega p e^2}{\omega^2} k^2 e^2 \left(1 - \frac{\omega p e^2}{\omega^2}\right) = 0 \]

1. When \( \omega > \omega p e \) (and \( \omega p e > \omega p i \)), \( \frac{\omega p e^2}{\omega^2} \ll 1 \), so
   \[ \omega^2 - \omega p e^2 - k^2 e^2 = 0 \]  \text{High Frequency Limit}

2. When \( \omega < \omega p i \) (and \( \omega p e > \omega p i \)), \( \omega^2 \ll \omega p e^2 \), so
   \[ -\omega p e^2 - k^2 e^2 \left(1 - \frac{\omega p i^2}{\omega^2}\right) = 0 \]  \text{Low Frequency Limit}

E. Langmuir Waves
   High Frequency, \( \omega > \omega p e \)

1. \( \omega^2 = \omega p e^2 + k^2 e^2 \)

2. \text{NOTE: } \frac{\omega p e^2}{\omega^2} \lambda_{De}^2 = \left(\frac{e^2 q e^2}{2 m e}\right)(\frac{k e^2}{A e^2 q e^2}) = \frac{k e}{m e} = Ce^2
   \text{Thus } \omega^2 = \omega p e^2 \left(1 + k^2 \lambda_{De}^2\right)

3. Wavelength Limits:
   a. Long wavelength: \( k^2 \lambda_{De}^2 \ll 1 \) \( \Rightarrow \frac{Ce^2}{k^2} \ll 1 \) or \( k \gg \lambda_{De} \)
      \text{(Non-propagating)} \( \omega^2 = \omega p e^2 \)
      Usual electron plasma oscillators as in cold plasma theory
   b. Short wavelength: \( k^2 \lambda_{De}^2 \gg 1 \) \( \text{or } k \ll \lambda_{De} \)
      \( \omega^2 = k^2 e^2 \) \text{Electron acoustic wave (Non-dispersive)
5. **IMPORTANT NOTE:** At wavelengths such that $k \lambda_e > 1$, the phase velocity of the waves $V_p = \frac{c_0}{k}$ is very near the electron thermal velocity $V_{te} = \sqrt{\frac{2kT_e}{m_e}}$.

b. The waves are resonant with the electrons. This leads to strong collisionless damping of the electron acoustics wave called **Landau damping**.

c. Thus, electron acoustics waves do not really occur.

d. This is a failure of the two-fluid theory.

⇒ **Kinetic theory is required to correctly describe collisionless damping.**

**F. Ion Acoustic Waves:** Low Frequency, $\omega < \omega_{pi}$

\[ -\omega^2 - \frac{k^2c_e^2}{a_{pe}^2} \left( 1 - \frac{\omega^2}{\omega_{pi}^2} \right) = 0 \]

\[ \Rightarrow \omega^2 = \frac{\omega_{pi}^2 k^2 c_e^2}{a_{pe}^2 + k^2 c_e^2} = \frac{\omega_{pi}^2}{a_{pe}^2} \left( 1 + \frac{k^2 c_e^2}{a_{pe}^2} \right) \]

\[ a. \text{ NOTE: } a_{pe}^2 = \frac{(\frac{m_e}{e})^2}{\left( \frac{m_e + m_i}{e} \right)^2} = \frac{m_e}{m_i} \Rightarrow \omega^2 = \frac{k^2 c_i^2}{1 + k^2 c_i^2} \]

\[ b. \text{ Ion Acoustic Dispersion Relation } \omega^2 = \frac{k^2 c_i^2}{1 + k^2 c_i^2} \]

\[ c. \text{ DEF: Ion Acoustic Speed } c_i = \frac{kT_i}{m_i} \text{ - Electrons provide pressure, Ions provide inertia.} \]
2. Wavelength Limits:
   a. Long wavelength: \( k^2 \lambda_{De}^2 \ll 1 \)
   \[
   \omega^2 = k^2 c_i^2 \quad \text{Ion Acoustic Wave (Non-dispersive)}
   \]
   \[\frac{\omega}{k} = \pm c_i\]
   
   b. Short Wavelength: \( k^2 \lambda_{De}^2 \gg 1 \)
   \[
   \omega^2 = \frac{k^2 c_i^2}{k^2 \lambda_{De}^2} = \frac{(kTe/m_i)}{(6\sqrt{2} \pi \epsilon_0 e^2/c_0 m_i)} = \frac{N_i q_i^2}{c_0 m_i} = \frac{N_i q_i^2}{c_0 m_i} = \omega p_i^2
   \]
   \[
   \omega^2 = \omega p_i^2 \quad \text{Plasma oscillations of ion plasma frequency.}
   \]

3a. If \( T_i = T_e \) (instead of \( T_i \ll T_e \) as we assumed),
   then the ion acoustic phase speed \( \frac{\omega}{k} = c_i = \sqrt{\frac{kTe}{m_i}} \approx \sqrt{\frac{2kT_i}{m_i}} = V_{ti} \)

b. Thus, when ion temperature is equal to electron temperature, ions' thermal velocity is resonant with the wave phase speed.

c. Ion Acoustic waves experience strong collisionless Landau damping when \( T_i = T_e \) (Again, kinetic description is necessary)

d. Only when \( T_i \ll T_e \) do ion acoustic waves propagate with little damping.