

Lecture #4 - Multiple Timescale Methods

Howes ①

I. Application of Multiple Timescale Methods

A. We'll see how to apply this powerful analytical method with a simple example first.

B. Example - Duffing's Equation

1. A simple nonlinear oscillator problem is given by

$$\text{Duffing's Equation} \quad \frac{d^2x}{dt^2} = -x + x^3$$

(For more info, see

<http://mathworld.wolfram.com/DuffingDifferentialEquation.html>)

2. To solve this problem, we will assume the system evolves on two,

separable timescales: Short t

Long $\tau = \epsilon^2 t$

b. We will treat these as separate variables.

c. Here $\epsilon \ll 1$ is a small dimensionless number, used for bookkeeping.

d. NOTE: $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon^2 \frac{\partial^2}{\partial t \partial \tau} + \epsilon^4 \frac{\partial^2}{\partial \tau^2}$$

3. ~~As~~ usual with the simple harmonic oscillator, we'll make the assumption of small amplitude oscillations.

Expand Solution ~~xxxxxx~~ $x(t, \tau) = \epsilon x_1(t, \tau) + \epsilon^2 x_2(t, \tau) + \epsilon^3 x_3(t, \tau) + \dots$

4. Plug expansion for x and $\frac{d}{dt}$ into original equation:

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} (\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + 2\epsilon^2 \frac{\partial^2}{\partial t \partial \tau} (\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + \epsilon^4 \frac{\partial^2}{\partial \tau^2} (\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) \\ & = -(\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3) + (\epsilon^3 x_1^3 + 3\epsilon^4 x_1^2 x_2 + 3\epsilon^5 x_1 x_2^2 + 3\epsilon^5 x_1^2 x_3 + \dots) \end{aligned}$$

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5. Find equations at each power of ϵ :

a. $\mathcal{O}(\epsilon)$: $\frac{\partial^2 x_1}{\partial t^2} = -x_1$

b. $\mathcal{O}(\epsilon^2)$: $\frac{\partial^2 x_2}{\partial t^2} = -x_2$

c. $\mathcal{O}(\epsilon^3)$: $\frac{\partial^2 x_3}{\partial t^2} + 2 \frac{\partial^2 x_1}{\partial t \partial \tau} = -x_3 + x_1^3$

6. Solve $\mathcal{O}(\epsilon)$ equation:

a. General solution for $x_1(t, \tau)$: $x_1(t, \tau) = A(\tau) \cos t + B(\tau) \sin t$

b. On the slow timescale τ , $A(\tau)$ and $B(\tau)$ are treated as constants. The higher order equations will allow us to solve for $A(\tau)$, $B(\tau)$

7. $\mathcal{O}(\epsilon^2)$ equation does not tell us anything new.

8. Solve $\mathcal{O}(\epsilon^3)$ equation:

a. We have solved for x_1 , so we can substitute in:

NOTE: $\frac{\partial^2 x_1}{\partial t \partial \tau} = -\frac{\partial A}{\partial \tau} \sin t + \frac{\partial B}{\partial \tau} \cos t$

b. Thus: $\frac{\partial^2 x_3}{\partial t^2} + x_3 = -2 \frac{\partial A}{\partial \tau} \sin t + 2 \frac{\partial B}{\partial \tau} \cos t + A^3 \cos^3 t + 3A^2 B \cos^2 t \sin t + 3AB^2 \cos t \sin^2 t + B^3 \sin^3 t$

c. i. We assume the x_3 is periodic over one oscillation $[0, 2\pi]$

ii. Therefore, we can annihilate x_3 by averaging over an oscillation.

d. TRICK: Multiply the equation by $\sin t$ and integrate $\int_0^{2\pi} dt$

i. LHS: $\int_0^{2\pi} \sin t \left(\frac{\partial^2 x_3}{\partial t^2} + x_3 \right) dt$

ii. Integrate by parts twice on $\frac{\partial^2 x_3}{\partial t^2}$ first term:

$$\int_0^{2\pi} \sin t \frac{\partial^2 x_3}{\partial t^2} dt = \cancel{\sin t \frac{\partial x_3}{\partial t}} \Big|_0^{2\pi} - \cancel{\cos t x_3} \Big|_0^{2\pi} - \int_0^{2\pi} \sin t x_3 dt$$

by periodicity of x_3

iii. Thus $\int_0^{2\pi} \sin t (-x_3 + x_3) dt = 0$

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$$2. \text{ RHS: } 2 \frac{\partial A}{\partial T} \int_0^{2\pi} \sin^2 t \, dt = 2\pi \frac{\partial A}{\partial T}$$

$$-2 \frac{\partial B}{\partial T} \int_0^{2\pi} \sin t \cos t \, dt = -2 \frac{\partial B}{\partial T} \left[\frac{\sin^2 t}{2} \right]_0^{2\pi} = 0$$

$$A^3 \int_0^{2\pi} \sin t \cos^3 t \, dt = A^3 \left[-\frac{\cos^4 t}{4} \right]_0^{2\pi} = 0$$

$$3A^2B \int_0^{2\pi} \cos^2 t \sin^2 t \, dt = 3AB \int_0^{2\pi} (\sin^2 t - \sin^4 t) \, dt = 3A^2B \left(\pi - \frac{3\pi}{4} \right)$$

$$3AB^2 \int_0^{2\pi} \sin^3 t \cos t \, dt = 3AB^2 \left[\frac{\sin^4 t}{4} \right]_0^{2\pi} = 0 \quad = \frac{3\pi}{4} A^2 B$$

$$B^3 \int_0^{2\pi} \sin^4 t \, dt = \frac{3\pi}{4} B^3$$

NOTE: 1. $\int_0^{2\pi} \sin^2 t \, dt = \pi$

2. $\int_0^{2\pi} \sin^4 t \, dt = \frac{3\pi}{4}$

3. Thus RHS = $2\pi \frac{\partial A}{\partial T} + \frac{3\pi}{4} A^2 B + \frac{3\pi}{4} B^3$

4. Thus

$$\frac{\partial A}{\partial T} = -\frac{3}{8} (A^2 B + B^3) = -\frac{3}{8} (A^2 + B^2) B$$

e. We can perform the same trick, this time multiplying by $\cos t$ and $\int_0^{2\pi} dt$.

1. Again LHS = 0

2. RHS = $-2\pi \frac{\partial B}{\partial T} + \frac{3\pi}{4} A^3 + \frac{3\pi}{4} AB^2$

3. Thus

$$\frac{\partial B}{\partial T} = \frac{3}{8} (A^3 + AB^2) = \frac{3}{8} (A^2 + B^2) A$$

f. Thus, we have

$$\boxed{\begin{aligned} \frac{\partial A}{\partial T} &= -\frac{3}{8} (A^2 + B^2) B \\ \frac{\partial B}{\partial T} &= \frac{3}{8} (A^2 + B^2) A \end{aligned}}$$

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g. These equations for $A(\tau)$ & $B(\tau)$ are also nonlinear, but they may be solved by another trick.

1. TRICK: Multiply $\frac{\partial A}{\partial \tau}$ by A , $\frac{\partial B}{\partial \tau}$ by B and add equations

$$2. A \frac{\partial A}{\partial \tau} + B \frac{\partial B}{\partial \tau} = \frac{1}{2} \frac{\partial}{\partial \tau} (A^2 + B^2) = 0$$

3. Therefore $A^2 + B^2 = \text{constant}$. Let $C^2 \equiv A^2 + B^2$

$$h. \text{ Thus } \frac{\partial A}{\partial \tau} = -\frac{3C^2}{8} B \quad \frac{\partial B}{\partial \tau} = \frac{3C^2}{8} A$$

$$1. \text{ Solving } \frac{\partial^2 A}{\partial \tau^2} = -\frac{3C^2}{8} \frac{\partial B}{\partial \tau} = -\left(\frac{3C^2}{8}\right)^2 A$$

2. General solution can be written

$$A(\tau) = A_0 \cos\left(\frac{3C^2}{8}\tau + \phi_0\right)$$

where A_0 is amplitude
 ϕ_0 is phase

$$3. \text{ Thus } B(\tau) = A_0 \sin\left(\frac{3C^2}{8}\tau + \phi_0\right)$$

i. Plugging in $A(\tau)$ and $B(\tau)$ to get full x_1 solution:

$$x_1(t, \tau) = A_0 \cos\left(\frac{3C^2}{8}\tau + \phi_0\right) \cos t + A_0 \sin\left(\frac{3C^2}{8}\tau + \phi_0\right) \sin t$$

$$x_1(t, \tau) = A_0 \cos\left(t - \frac{3C^2}{8}\tau + \phi_0\right)$$

9. Let's check our assumption that x_3 is periodic over oscillation in t .

$$a. \frac{\partial^2 x_3}{\partial t^2} + x_3 = -2 \frac{\partial^2 x_1}{\partial t \partial \tau} + x_1^3 \quad \text{This is a driven harmonic oscillator.$$

$$b. \frac{\partial^2 x_1}{\partial t \partial \tau} = \frac{\partial}{\partial \tau} \left[-A_0 \sin\left(t - \frac{3C^2}{8}\tau + \phi_0\right) \right] = \frac{3C^2 A_0}{8} \cos\left(t - \frac{3C^2}{8}\tau + \phi_0\right)$$

$$c. \text{ Thus, } \frac{\partial^2 x_3}{\partial t^2} + x_3 = -\frac{6C^2 A_0}{8} \cos\left(t - \frac{3C^2}{8}\tau + \phi_0\right) + A_0^3 \cos^3\left(t - \frac{3C^2}{8}\tau + \phi_0\right)$$

d. Since forcing term (RHS) is periodic in t , so must be solution $x_3(t, \tau)$.

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10. Final Solution.

a. NOTE: $C^2 = A^2 + B^2 = A_0^2$

b. See our bookkeeping term $\epsilon = 1$.

$$x_1(t) = A_0 \cos\left(t - \frac{3A_0^2}{8}t + \phi_0\right)$$

c. Small amplitude assumption means $t \gg \frac{3A_0^2}{8}t$

11. Interpretation: a. System is, to lowest order, a harmonic oscillator

b. Over long times, $x_1(t)$ builds up a large phase shift $-\frac{3A_0^2}{8}t$ due to the nonlinear term.

II. Dimensionless Equations

A. Conversion of equations to a dimensionless form often leads to simpler less cumbersome relations when working with the equations.

B. Example: Linearized Hydrodynamic Momentum Equation

1. $\rho_0 \frac{\partial u}{\partial t} = -\nabla p$

2. Choose characteristic values by which to normalize:

a. $u' = \frac{u}{c_s}$ $p' = \frac{p}{p_0}$

b. $\nabla = \frac{\partial}{\partial x}$, so $x' = \frac{x}{L_0}$ $t' = \frac{t}{\left(\frac{L_0}{c_s}\right)} = \frac{c_s t}{L_0}$

3. Substitute normalized values into equation

a. $\rho_0 \frac{\partial (u' c_s)}{\partial \left(t' \frac{L_0}{c_s}\right)} = - \frac{\partial (p' p_0)}{\partial (x' L_0)} \Rightarrow \frac{c_s^2}{L_0} \frac{\partial u'}{\partial t'} = - \left(\frac{p_0}{\rho_0}\right) \frac{1}{L_0} - \nabla' p'$

b. NOTE: Speed of sound $c_s^2 \equiv \left(\frac{p_0}{\rho_0}\right)$, so this leaves

$$\frac{\partial u'}{\partial t'} = -\nabla' p'$$