Lecture 5: Magnetic Moment and the Mirror Force

I. Magnetic Moment
   A. Magnetic moment due to particle Larmor Motion
      1. A current loop has a magnetic moment $\mathbf{M} = I \mathbf{A}$
      2. For a charged particle with charge $q$ in Larmor Motion
         $I = \frac{\text{charge}}{\text{Time}} = \frac{q}{(2\pi a c)} = \frac{d\mathbf{v}}{dt}$
         $A = \pi r_0^2 \approx \pi \frac{V_{1,2}^2}{a c^2}$
         $\mathbf{M} = I \mathbf{A} = \left( \frac{q}{2\pi} \right) \left( \pi \frac{V_{1,2}^2}{a c^2} \right) = \frac{q V_{1,2}^2}{2(\frac{a c}{m})}$
         $\frac{m v_{1,2}^2}{2B} = \mathbf{M}$

II. The Mirror Force
   A. What happens when $\nabla \times \mathbf{B} \parallel \mathbf{B}$?
      1. Because Maxwell's Equations demand $\nabla \cdot \mathbf{B} = 0$, for magnetic field to increase along field line, another must change.
         a. Cylindrical Coordinates $\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial}{\partial r} (r \mathbf{B}_r) + \frac{1}{r} \frac{\partial \mathbf{B}_\phi}{\partial \phi} + \frac{\partial \mathbf{B}_z}{\partial z} = 0$
         b. Take azimuthally symmetric field ($\mathbf{B}_\phi = 0$) with no azimuthal component ($\mathbf{B}_\phi = 0$),
            $\frac{1}{r} \frac{\partial}{\partial r} (r \mathbf{B}_r) = -\frac{\partial \mathbf{B}_z}{\partial z}$
         c. Assuming $\frac{\partial \mathbf{B}_z}{\partial z}$ is independent of $r$ (valid for small $r$),
            we can integrate to yield,
            $\mathbf{B}_r = -\frac{L}{2} \frac{\partial \mathbf{B}_z}{\partial z}$ (Assume constant of integration is zero)
   2. Increasing field along z direction requires a $\mathbf{B}_r$ component.
   3. What is the particle motion in such a field?
II. Force on a Particle

1. We want to find \( \mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \) for this case.
   
   a. \( \mathbf{B} = e \mathbf{B}_r \hat{r} + e \mathbf{B}_z \hat{z} = -e \frac{r \partial B_z}{2} \hat{r} + B_z \hat{z} \)
   
   b. \( \mathbf{V} = v_r \hat{r} + v_\theta \hat{\theta} + v_z \hat{z} \)

2. In cylindrical coordinates:
   \[
   \mathbf{F} = q \left( v_\theta \mathbf{B}_z - v_z \mathbf{B}_\theta \right) \hat{\theta} + q \left( e v_r \mathbf{B}_r - v_\theta \mathbf{B}_z \right) \hat{r} + q \left( v_r \mathbf{B}_\theta - e v_\theta \mathbf{B}_r \right) \hat{\theta}
   \]

3. Order of Magnitude:
   \[
   \mathbf{F} = q v_\theta \mathbf{B}_z \hat{r} - q v_r \mathbf{B}_z \hat{\theta} = q \left( v_\theta \hat{r} + v_r \hat{\theta} \right) \times \mathbf{B}_z
   \]
   a. These terms are just the usual terms dictating Larmor motion.
   b. This is easy to see for a particle with a guiding center at \( Z = 0 \).

   \[
   \hat{r} \rightarrow \mathbf{B} \rightarrow \hat{\theta} \rightarrow \mathbf{F} \rightarrow \text{For this case, } \mathbf{v}_\theta = -v_L \text{, } \mathbf{v}_r = 0 \text{ (for } q > 0) \]
   c. Thus \( \mathbf{F} = -q v_L \mathbf{B}_z \hat{r} \) provides centripetal acceleration
      for Larmor motion.
   d. \( \frac{mv^2}{r} = q v_L B_z \) Solving for \( r = \frac{m v_L}{q B} = \frac{v_L}{v_c} = r_L \)

4. One axial component: \( \mathbf{F}_z = -q v_\phi \mathbf{B}_r \)
   a. Again, we take the case for a particle with guiding center at \( Z = 0 \).
      This \( \mathbf{v}_\phi = -v_L \) and \( r = r_L \).
   b. \[
   \mathbf{F}_z = -q \left( v_L \right) \left( -\frac{2 \mu}{2} \frac{\partial B_z}{\partial z} \right) = -q \left( \frac{v_L^2}{2 v_c} \frac{\partial B_z}{\partial z} \right) = -\left( \frac{mv^2}{2B} \right) \frac{\partial B_z}{\partial z} = \left( \frac{mv^2}{2B} \right) \frac{\partial B_z}{\partial z}
   \]
   \( r = \frac{\mu}{v_c} \)

   This can be written \( \mathbf{F}_z = \mu \frac{\partial B_z}{\partial z} \text{ Magnetic Mirror Force} \)
5. The mirror force accelerates the particle along the field line in the direction of decreasing magnetic field magnitude.

6. This can be written in general as:

\[ F = - \mu (\hat{\mathbf{b}} \cdot \nabla) \mathbf{B} \]

where \( \hat{\mathbf{b}} \cdot \nabla \) is the gradient along the field \( \mathbf{B} \).

a. Compare to the electrostatic force on a charge.

For \( \mathbf{E} = -\nabla \phi \) and \( \mathbf{F} = q \mathbf{E} \), \( F = -q \nabla \phi \).

b. The mirror force acts on the particle magnetic moment \( \mu = \frac{mv^2}{2EB} \), where the field magnitude \( B \) appears like a potential.

\[ \Rightarrow \text{Repels particles from strong field region!} \]

7. \( \mathbf{E} \times \mathbf{B} \): Azimuthal Component \( F_\phi = q v_z B_r \)

a. The presence of an azimuthal component of force means particles can gain energy in the perpendicular component at rate \( V_p F_b \).

b. For perpendicular energy \( v_\perp = \frac{1}{2} m v_z^2 \), we have

\[ \frac{dv_\perp}{dt} = V_p F_b = q (v_\phi) v_z B_r \]

c. For ions, \( V_\phi = -v_\perp \) and \( B_r = -\frac{v_\perp}{v_z} \frac{dB_z}{dz} \) for particle guiding center \( z = 0 \).

\[ \frac{dv_\perp}{dt} = q (-v_\perp) \left( -\frac{v_\perp}{v_z} \frac{dB_z}{dz} \right) v_z = \frac{q v_\perp}{2v_z} v_z \frac{dB_z}{dz} = \frac{mv_z^2 v_\perp^2}{2B} \frac{dB_z}{dz} = \mu \frac{v_\perp}{v_z} \frac{dB_z}{dz} \]

d. Note: \( \frac{dB}{dt} = \frac{d^2 B}{d^2 t} + \frac{d}{dx} \frac{dB}{dx} + \frac{d}{dy} \frac{dB}{dy} + \frac{d}{dz} \frac{dB}{dz} = \frac{d}{dx} \frac{dB}{dx} + \frac{d}{dy} \frac{dB}{dy} + \frac{d}{dz} \frac{dB}{dz} = \mu \frac{v_\perp}{v_z} \frac{dB_z}{dz} \)

Since \( \frac{dB_z}{dz} = v_z \frac{dB_z}{dz} \) at \( z = 0 \), we get

\[ \frac{dv_\perp}{dt} = \mu \frac{dB_z}{dz} \]

\[ \text{But } \mu = \frac{w_i}{B}, \text{ so } \frac{1}{B} \frac{dv_\perp}{dt} = \frac{w_i}{B^2} \frac{dB_z}{dz} = 0 \Rightarrow \frac{d(w_i)}{dt} = 0 \Rightarrow \frac{d}{dt} = 0 \]

\[ \text{Hawes}\]
III. Adiabatic Invariance

A. Interpretation:
1. \( \frac{d\mu}{dt} = 0 \) implies that, as a charged particle moves through a changing field \( \mu = \frac{mv_i^2}{2B} \) remains constant.

B. Alternative Derivation:
1. First, note \( m\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{d}{dt}(\frac{1}{2}mv^2) = 2\vec{v} \cdot (\dot{v} \times \vec{B}) = 0 \)
   a. Therefore, total energy is constant \( E = \frac{1}{2}mv^2 = \frac{1}{2}m(v_{in}^2 + v_{li}^2) \)
2. Mirror Force equation: \( F_{li} = -\mu(v_{li}^2) \vec{v} \cdot \vec{B} = m \frac{dv_{li}}{dt} \)
   a. Multiply by \( v_{li} \):
     \( m v_{li} \frac{dv_{li}}{dt} = \frac{d}{dt}(\frac{1}{2}mv_{li}^2) = -\mu(v_{li}^2) \vec{v} \cdot \vec{B} = -\mu v_{li}^2 \vec{v} \cdot \vec{B} \)
   b. Again \( \frac{d\vec{B}}{dt} = \frac{d}{dt}(\frac{v_{li}}{B}) = \vec{v} \cdot \vec{B} = x_{li} \vec{B} \) so
     \( \frac{d}{dt}(\frac{1}{2}mv_{li}^2) = -\mu \frac{d\vec{B}}{dt} = -\mu \frac{1}{2B} \frac{d\vec{B}}{dt} = k = \frac{d}{dt}(\frac{mv_{li}^2}{2}) \)
   c. Note:
     \( \frac{d}{dt}(\frac{mv_{li}^2}{2}) = \frac{d}{dt}(\frac{mv_{li}^2}{2B}) = \frac{mv_{li}^2}{2B} \frac{1}{B} \frac{d\vec{B}}{dt} = 0 \)
   d. Note: \( \frac{d\mu}{dt} = \frac{d}{dt}(\frac{mv_{li}^2}{2B}) = \frac{1}{B} \frac{d(mv_{li}^2)}{dt} = -\frac{mv_{li}^2}{2B} \frac{1}{B} \frac{d\vec{B}}{dt} \)
   e. Thus \( \frac{d\mu}{dt} = 0 \).
IV. Confineance by Magnetic Mirror

A. Magnetic Mirror Machine:

1. Particles are confined by a magnetic mirror force at either end of the machine.

2. Pitch Angle: $\alpha = \text{angle between velocity vector and magnetic field.}$

\[ V_{\perp} = V \cos \alpha \]
\[ V_{\parallel} = V \sin \alpha \]

3. Parallel Equation of motion $F_{\parallel} = -\mu \mathbf{B} \cdot \mathbf{V} \mathbf{B}$ can be written:

\[ m \frac{dv_{\parallel}}{dt} = -\mu \frac{d\mathbf{B}}{ds} \] where $s$ is distance along field line.

a. $\frac{dv_{\parallel}}{dt} = \frac{d}{dt} V_{\parallel} = V_{\parallel} \frac{dv_{\parallel}}{ds}$ along field line.

b. $m v_{\parallel} \frac{dv_{\parallel}}{ds} = 2 \left( \frac{1}{2} m v_{\parallel}^2 \right) = -\mu \frac{d\mathbf{B}}{ds} = -\frac{d\mu B}{ds}$ since $\mu$ = constant.

c. Thus $\frac{d}{ds} \left( \frac{1}{2} m v_{\parallel}^2 + \mu B \right) = 0$ along field line

Therefore $E = \frac{1}{2} m v_{\parallel}^2(s) + \mu B(s)$

$E =$ const Conservation of Energy
$\mu =$ const Adiabatic Invariant
Legend #6 (continued)

II. A. (continued)

4. Potential Interpretation:
   a. A charged particle in an electrostatic field \( E = -\nabla \phi \)
      has conserved energy \( E = \frac{1}{2}mv^2 + q\phi \)
   b. Here conservation involves parallel velocity \( E = \frac{1}{2}mv_{||}^2 + mB \)
      and magnetic field magnitude

5. We can solve for \( v_{||} (s) \)

\[
v_{||}(s) = \pm \sqrt{\frac{2}{m} (E - mB(s))}
\]

where \( E \) and \( m \) are constants

a. When \( v_{||}(s_0) = 0 \), the particle reaches a turning point.
   Here \( B(s_0) = \frac{M}{E} \equiv B_+ \)
   Thus \( E = \frac{1}{2}mv_{||}^2 + mB = mB_+ \)

6. Physical Interpretation:

a. The particle experiences a changing \( B \) as it moves along field.
   b. Induced azimuthal force \( F_\phi \) does work on particle, increasing \( v_{||} \)
   c. Total energy \( \frac{1}{2}mv_{||}^2 + \frac{1}{2}mv^2 = E \)
      is conserved, so \( v_{||} \) must decrease
   d. Even \( v_{||} = 0 \), so the particle turns
      around, having been "mirrored"

7. How does pitch angle \( \alpha(s) \) change? \( E = \frac{1}{2}mv_{||}^2 + mB \) and \( E = \frac{1}{2}mv^2 \)
   a. \( v_{||} = v \cos \alpha \), so \( E = \frac{1}{2}mv^2 \cos^2 \alpha + mB = \frac{1}{2}mv^2 + mB \)
      Thus \( 1 - \cos^2 \alpha = \frac{mB}{E} \), or \( \sin^2 \alpha = \frac{mB}{E} = \frac{mB}{mB_+} = \frac{B}{B_+} \)
Lesson 16 (Continued)

V. A. 7. (Continued)

Thus \( \sin^2 \alpha(s) = \frac{B(s)}{B_t} \)

As \( B(s) \) increases to \( B_t \), \( \sin^2 \alpha(s) \to 1 \), or \( \alpha(s) \to \frac{\pi}{2} \).

8. Practical Considerations:

a. There is a limit to the maximum field strength:

\[
B_{\text{max}} \geq \mu B
\]

b. For particles with \( E > \mu B_{\text{max}} \), \( V_i = \sqrt{\frac{2E}{m} \left( E - \mu B \right)} > 0 \).

Thus, \( V_i \) never reaches zero \( \Rightarrow \) particles are not reflected.

c. Analogy: Frictionless ball on a hill/valley.

\[ E = \frac{1}{2}mv^2 + mg \]

i) If \( E > mg h_{\text{max}} \), ball possesses hill

ii) If \( E < mg h_{\text{max}} \), ball is trapped in valley, oscillating back & forth.

\[ \sin^2 \alpha_{\text{min}} = \frac{B_{\text{min}}}{B_{\text{max}}} \]

We know pitch angle \( \alpha \) increases as \( B \) increases. \( \sin^2 \alpha(s) = \frac{B(s)}{B_t} \)

Thus, at \( B = B_{\text{min}} \), pitch angle is at a minimum.

For a particle which reaches \( \alpha = \frac{\pi}{2} \) \( (V_i = 0) \) at \( B = B_{\text{max}} \), what is its pitch angle at \( B_0 \)?

For \( B(s) = B_{\text{min}} \),

\[ \sin^2 \alpha_{\text{min}} = \frac{B_{\text{min}}}{B_{\text{max}}} \]
e. This, for particles with $\alpha < \alpha_{\text{min}}$, particles will escape from magnetic mirror.

f. The Mirror Ratio $\frac{R_m}{\alpha_{\text{min}}}$

Thus $\sin^2 \alpha_{\text{min}} = \frac{1}{R_m}$

g. Looking in velocity space

Particles with $\alpha < \alpha_{\text{min}}$ will be lost.

h. In a collisionless plasma, all particles with $\alpha < \alpha_{\text{min}}$ will be lost from mirror.

i. In a collisionless plasma, particle collisions will scatter particles into the loss cone, and eventually much of the plasma will be lost.

8. Earth's Magnetosphere:

i. Dipole field of earth behaves as a magnetic mirror:

- Weak field at equator
- Strong field at poles

2. Particles trapped on field lines will bounce from pole to pole.

(Can't forget $\nabla B$ & curvature drifts also lead to motion westward around the earth.)