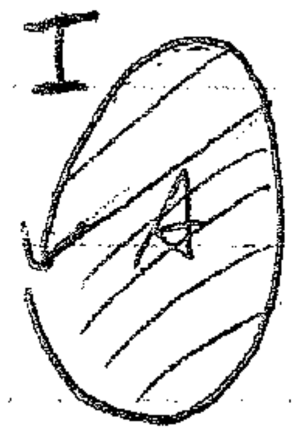
Lecture #6: Magnetic Moment and the Mirror Force

Hawes

I. Magnetic Moment

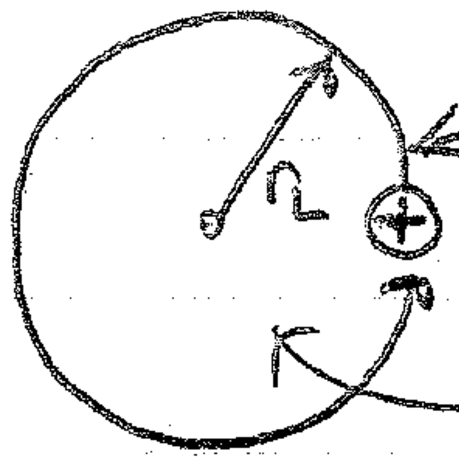
A. Magnetic moment due to particle Larmor Motion

1. A current loop has a magnetic moment $\mu = IA$



2. For a charged particle with charge q in Larmor Motion

a. $\underset{\sim}{B} \otimes$



$$I = \frac{\text{charge}}{\text{time}} = \frac{q}{(2\pi/\omega_c)} = \frac{\omega_c q}{2\pi}$$

$$A = \pi r_L^2 = \pi \frac{v_\perp^2}{\omega_c^2}$$

b. Thus $\mu = IA = \left(\frac{q\,\omega_c}{2\pi}\right)\left(\pi \frac{v_\perp^2}{\omega_c^2}\right) = \frac{q\, v_\perp^2}{2\left(\frac{qB}{m}\right)} = \boxed{\frac{m v_\perp^2}{2B} = \mu}$ Magnetic Moment

II. The Mirror Force

A. What happens when $\nabla B \parallel \underset{\sim}{B}$?

1. Because Maxwell's Equations demand $\nabla \cdot \underset{\sim}{B} = 0$, for magnetic field to increase along field line, another must change.

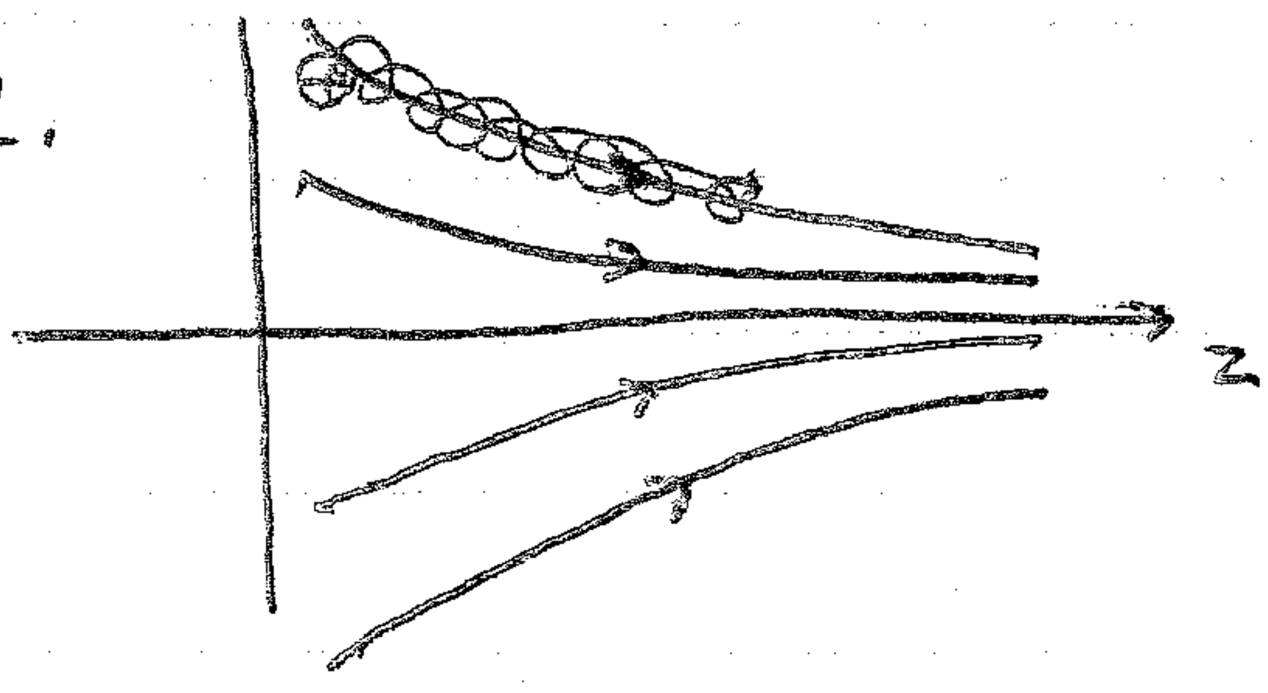
a. Cylindrical Coordinates (NRL p. 6) $\nabla \cdot \underset{\sim}{B} = \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0$

b. Take axisymmetric field $\left(\frac{\partial}{\partial \phi} = 0\right)$ with no azimuthal component $(B_\phi = 0)$.

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) = -\frac{\partial B_z}{\partial z}$$

c. Assuming $\frac{\partial B_z}{\partial z}$ is independent of r (valid for small r), we can integrate to yield. $B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}$ (Assume constant of integration is zero)

2.



Increasing field along z direction requires a B_r component.

3. What is the particle motion in such a field?

II. (Continued)

B. Force on Particle

1. We want to find $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ for this case.

   a. $\mathbf{B} = \epsilon B_r \hat{\mathbf{r}} + B_z \hat{\mathbf{z}} = -\epsilon \frac{r}{2}\frac{\partial B_z}{\partial z}\hat{\mathbf{r}} + B_z \hat{\mathbf{z}}$

   b. $\mathbf{v} = v_r \hat{\mathbf{r}} + v_\phi \hat{\boldsymbol{\phi}} + v_z \hat{\mathbf{z}}$

2. In cylindrical coordinates:

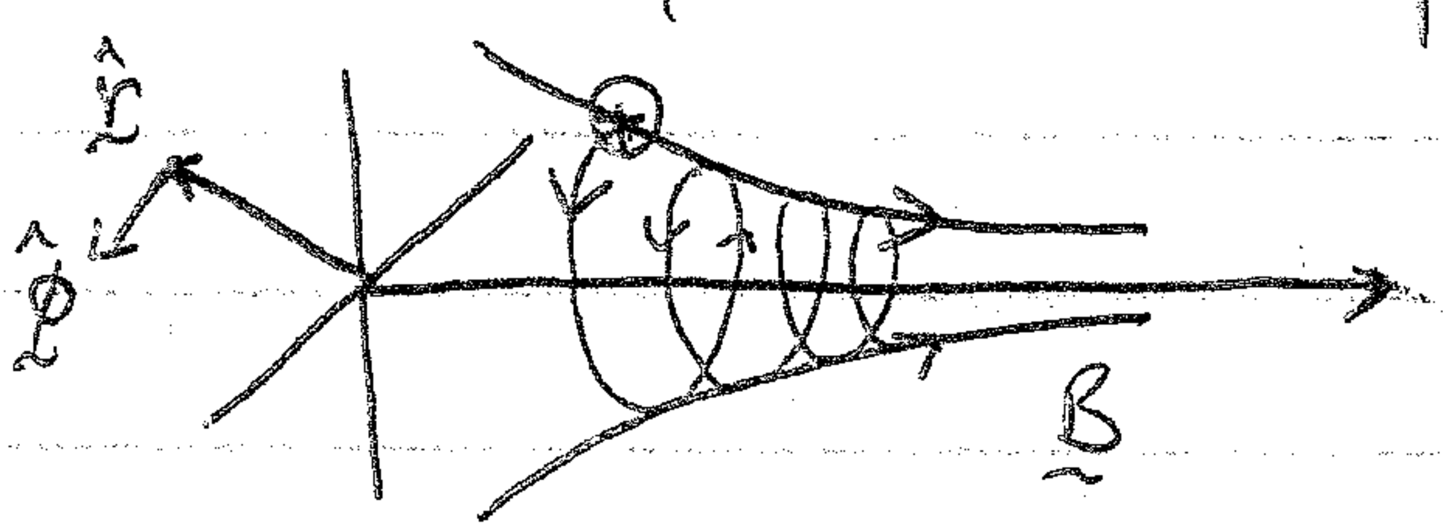
$$\mathbf{F} = \underbrace{q(v_\phi B_z - \cancel{v_z B_\phi})\hat{\mathbf{r}}}_{\text{large terms}} + q(\epsilon v_z B_r - v_r B_z)\hat{\boldsymbol{\phi}} + q(\cancel{v_r B_\phi}^{\,0} - \epsilon v_\phi B_r)\hat{\mathbf{z}}$$

large terms — small terms

3. $\mathcal{O}(1)$: $\mathbf{F} = q v_\phi B_z \hat{\mathbf{r}} - q v_r B_z \hat{\boldsymbol{\phi}} = q(v_\phi \hat{\boldsymbol{\phi}} + v_r \hat{\mathbf{r}}) \times (B_z \hat{\mathbf{z}})$

   a. These terms are just the usual terms dictating Larmor motion.

   b. This is easy to see for a particle with a <u>guiding center</u> at $z=0$.



For this case, $v_\phi = -v_\perp$ $\quad v_r = 0$
(for $q>0$)

   c. Thus $\mathbf{F} = -q v_\perp B_z \hat{\mathbf{r}}$ provides centripetal acceleration for Larmor motion.

   d. $\frac{m v_\perp^2}{r} = q v_\perp B_z$ Solving for $r = \frac{m v_\perp}{qB} = \frac{v_\perp}{\omega_c} = r_L$!

4. $\mathcal{O}(\epsilon)$: Axial Component: $F_z = -q v_\phi B_r$

   a. Again, we take the case for a particle guiding center at $z=0$. Thus $v_\phi = -v_\perp$ and $r = r_L$.

   b. $F_z = -q(-v_\perp)\left(\frac{-r_L}{2}\frac{\partial B_z}{\partial z}\right) \underset{\uparrow\; r_L = \frac{v_\perp}{\omega_c}}{=} -q \frac{v_\perp^2}{2\omega_c}\frac{\partial B_z}{\partial z} = -q\left(\frac{m}{qB}\right)\frac{v_\perp^2}{2}\frac{\partial B_z}{\partial z} = -\left(\frac{m v_\perp^2}{2B}\right)\frac{\partial B_z}{\partial z}$

This can be written $\boxed{F_z = -\mu \frac{\partial B_z}{\partial z}}$ Magnetic Mirror Force

II. B. (Continued)

5. The Mirror Force accelerates the particle <u>along the field line</u> in the direction of decreasing magnetic field magnitude.

6. This can be written in general as

$$\boxed{\underset{\sim}{F} = -\mu(\hat{\underset{\sim}{b}}\cdot\nabla)\underset{\sim}{B}}$$

where $\hat{b}\cdot\nabla$ is the gradient along the field $\underset{\sim}{B}$.

a. Compare to the electrostatic force on a charge.
For $\underset{\sim}{E} = -\nabla\phi$ and $\underset{\sim}{F} = q\underset{\sim}{E}$, $\underset{\sim}{F} = -q\nabla\phi$

b. The Mirror Force acts on the particle magnetic moment $\mu = \frac{mv_\perp^2}{2B}$, where the field magnitude B appears like a potential.
$\Rightarrow$ Repels particles from strong field region!

7. $\mathcal{O}(\epsilon)$: Azimuthal Component $F_\phi = q v_z B_r$

a. The presence of an azimuthal component of force means particles can gain energy in the perpendicular component at rate $v_\phi F_\phi$.

b. For perpendicular energy $W_\perp = \frac{1}{2}mv_\perp^2$, we have

$$\frac{dW_\perp}{dt} = v_\phi F_\phi = q(v_\phi) v_z B_r$$

c. For ions, $v_\phi = -v_\perp$ and $B_r = -\frac{r_L}{2}\frac{\partial B_z}{\partial z}$ for particle guiding center at $z=0$,

$$\frac{dW_\perp}{dt} = q(-v_\perp)\left(-\frac{r_L}{2}\frac{\partial B}{\partial z}\right)v_z = \frac{q\, v_\perp^2}{2\omega_c} v_z \frac{\partial B_z}{\partial z} = \frac{mv_\perp^2}{2B} v_z \frac{\partial B}{\partial z} = \mu\, v_z \frac{\partial B_z}{\partial z}$$

d. Note: $\frac{dB}{dt} = \frac{\partial B}{\partial t} + \frac{dx}{dt}\frac{\partial B}{\partial x} + \frac{dy}{dt}\frac{\partial B}{\partial y} + \frac{dz}{dt}\frac{\partial B}{\partial z} = \frac{\partial B}{\partial t} + v_x\frac{\partial B}{\partial x} + v_y\frac{\partial B}{\partial y} + v_z\frac{\partial B}{\partial z} = \frac{\partial B}{\partial t} + \underset{\sim}{v}\cdot\nabla B$

Since $\frac{dB_z}{dt} = \cancelto{0}{\frac{\partial B_z}{\partial t}} + \underset{\sim}{v}\cdot\nabla B_z = v_z\frac{\partial B}{\partial z}$, we get

$$\frac{dW_\perp}{dt} = \mu\frac{dB}{dt}$$

e. But $\mu = \frac{W_\perp}{B}$, so $\frac{1}{B}\frac{dW_\perp}{dt} - \frac{W_\perp}{B^2}\frac{dB}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\frac{W_\perp}{B}\right) = 0 \Rightarrow \boxed{\frac{d\mu}{dt} = 0}$

~~[illegible]~~

III. Adiabatic Invariance

A. Interpretation:

1. $\frac{d\mu}{dt} = 0$ implies that, as a charged particle moves through a changing field $\mu = \frac{m v_\perp^2}{2B}$ remains constant.

B. Alternative Derivation:

1. First, note $m\underset{\sim}{v} \cdot \frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\underset{\sim}{v} \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) = 0$

a. Therefore, total energy is constant $\mathcal{E} = \frac{1}{2}mv^2 = \frac{1}{2}m(v_\parallel^2 + v_\perp^2)$

b. Thus $\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) + \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{mv_\parallel^2}{2}\right) = -\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right)$

2. Mirror Force equation: $F_\parallel = -\mu(\hat{\underset{\sim}{b}} \cdot \nabla)B = m\frac{dv_\parallel}{dt}$

a. Multiply by $v_\parallel$:

$$m v_\parallel \frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu(v_\parallel \hat{\underset{\sim}{b}}) \cdot \nabla B = -\mu\, \underset{\sim}{v}_\parallel \cdot \nabla B$$

b. Again $\frac{dB}{dt} = \cancel{\frac{\partial B}{\partial t}}^{0} + \underset{\sim}{v} \cdot \nabla B = \underset{\sim}{v}_\parallel \cdot \nabla B$, so

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{dB}{dt} = \underbrace{-\frac{mv_\perp^2}{2}\frac{1}{B}\frac{dB}{dt} = -\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right)}$$

c. ~~Now:~~

Multiply by $\frac{1}{B}$

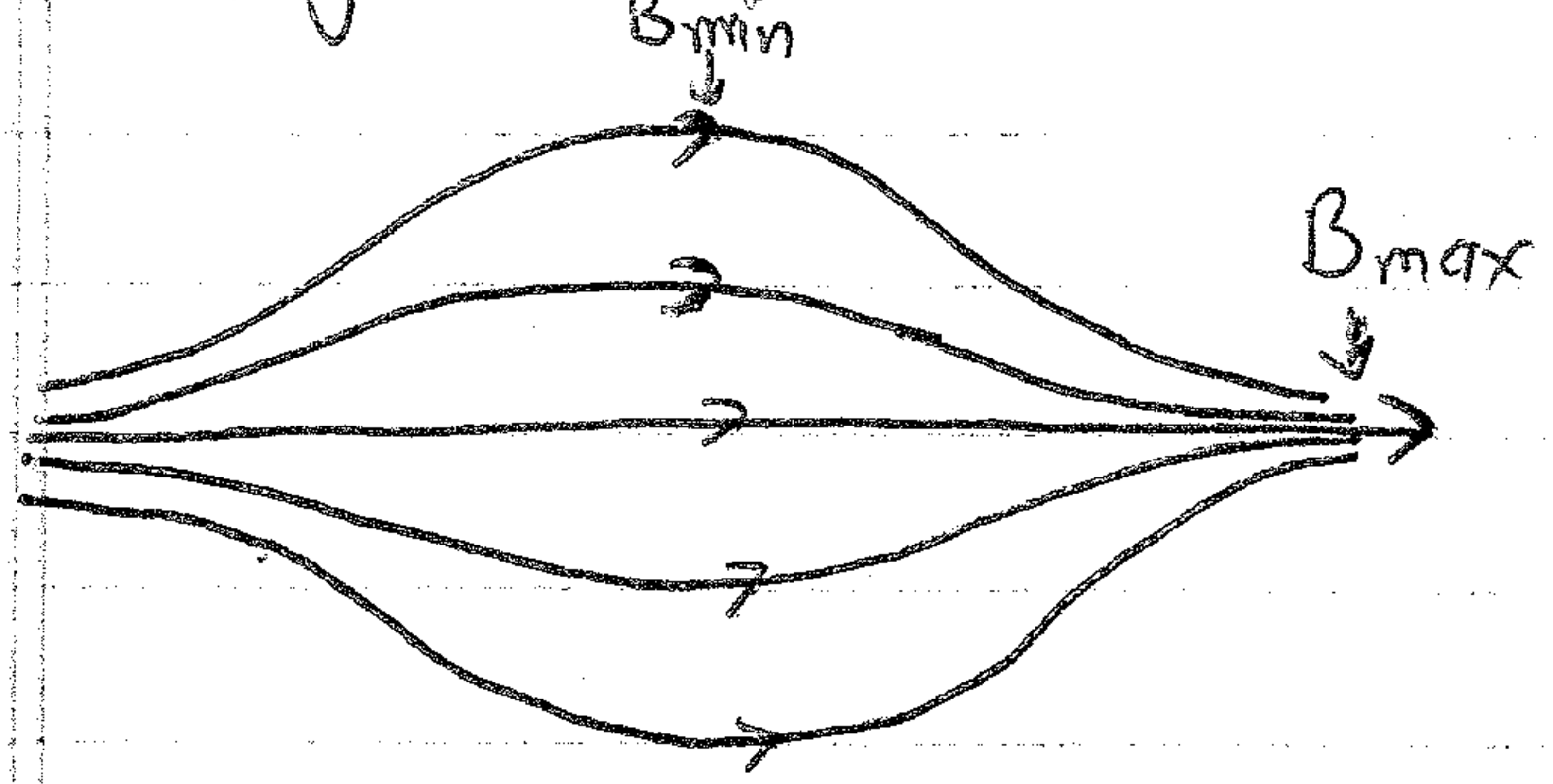
$$\frac{1}{B}\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right) - \frac{mv_\perp^2}{2}\frac{1}{B^2}\frac{dB}{dt} = 0$$

d. NOTE: $\frac{d\mu}{dt} = \frac{d}{dt}\left(\frac{\frac{mv_\perp^2}{2}}{B}\right) = \frac{1}{B}\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right) - \frac{mv_\perp^2}{2}\frac{1}{B^2}\frac{dB}{dt}$

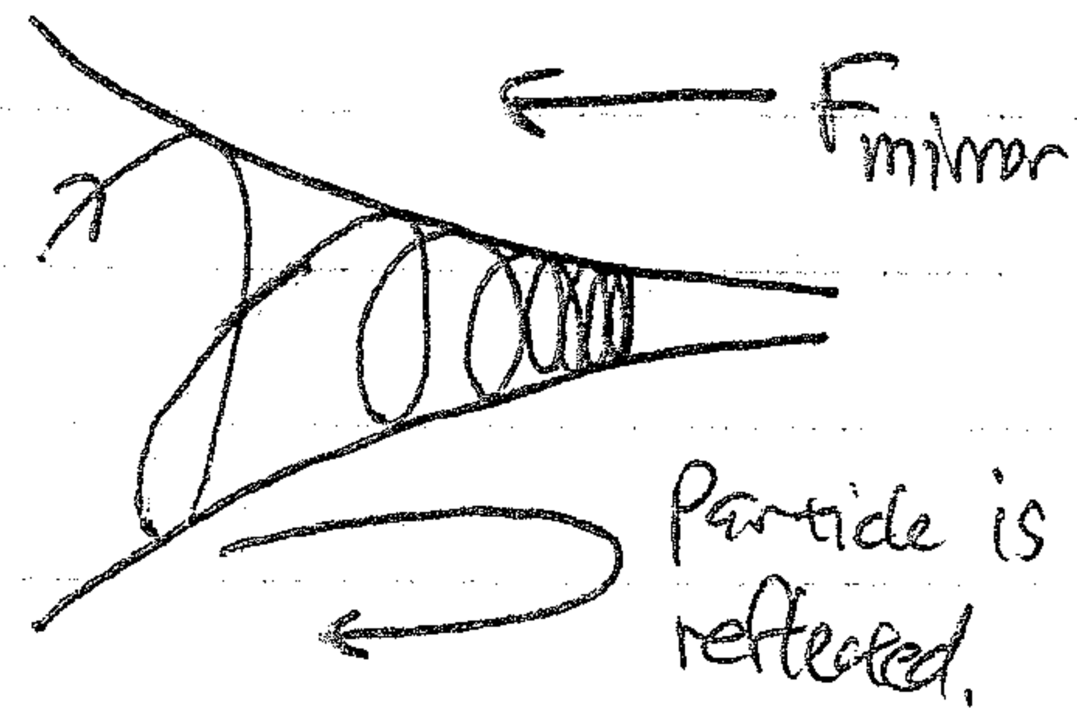
e. Thus $\boxed{\frac{d\mu}{dt} = 0.}$

IV. Confinement by Magnetic Mirror

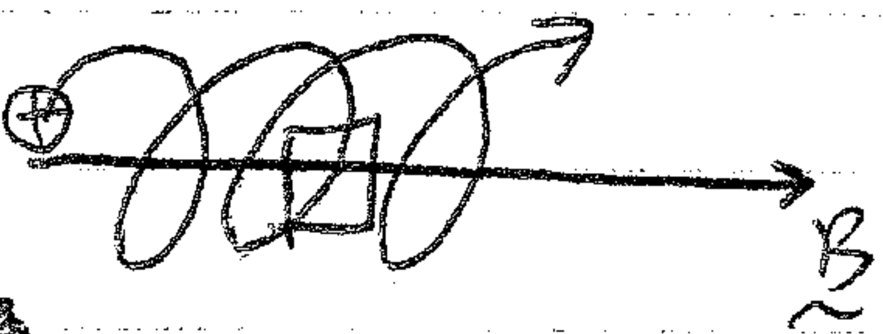
A. Magnetic Mirror Machine:

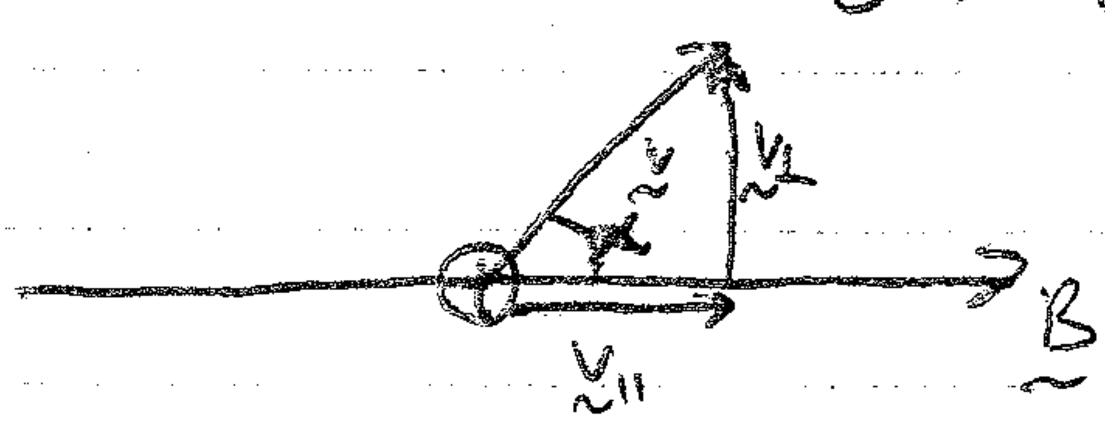


1. Particles are confined by magnetic mirror force at either end of the machine.

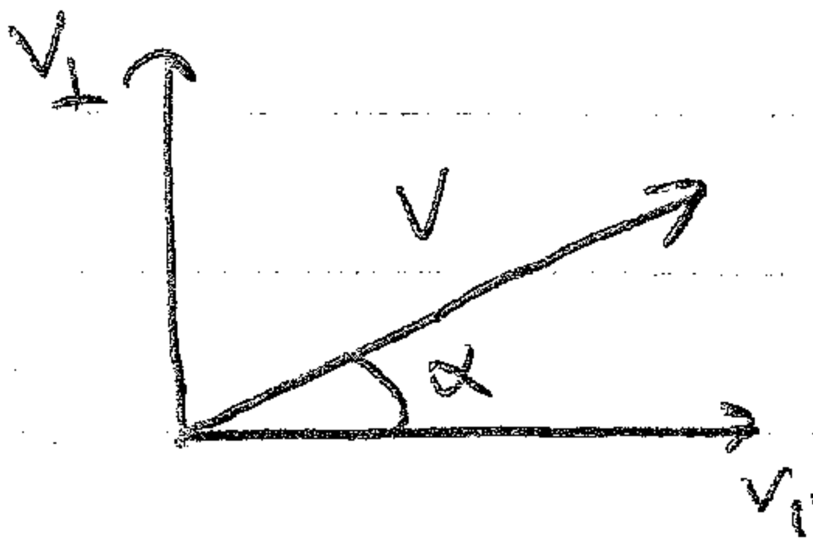


2. Pitch Angle:





$\alpha \equiv$ angle between velocity vector and magnetic field.



$v_\parallel = v\cos\alpha$

$v_\perp = v\sin\alpha$

3. Parallel Equation of motion $F_\parallel = -\mu\, \underset{\sim}{\hat{b}}\cdot\nabla B$ can be written

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial s}$$ where "s" is distance along field line.

a. $$\frac{dv_\parallel}{dt} = \cancel{\frac{\partial v_\parallel}{\partial t}}^{\,0} + \underset{\sim}{v}\cdot\nabla v_\parallel = v_\parallel\frac{\partial v_\parallel}{\partial s}$$ along field line.

b. $$m v_\parallel \frac{\partial v_\parallel}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{2}m v_\parallel^2\right) = -\mu\frac{\partial B}{\partial s} = -\frac{\partial\, \mu B}{\partial s}$$ since $\mu =$ constant.

c. Thus $$\frac{d}{ds}\left(\frac{1}{2}m v_\parallel^2 + \mu B\right) = 0$$ along field line

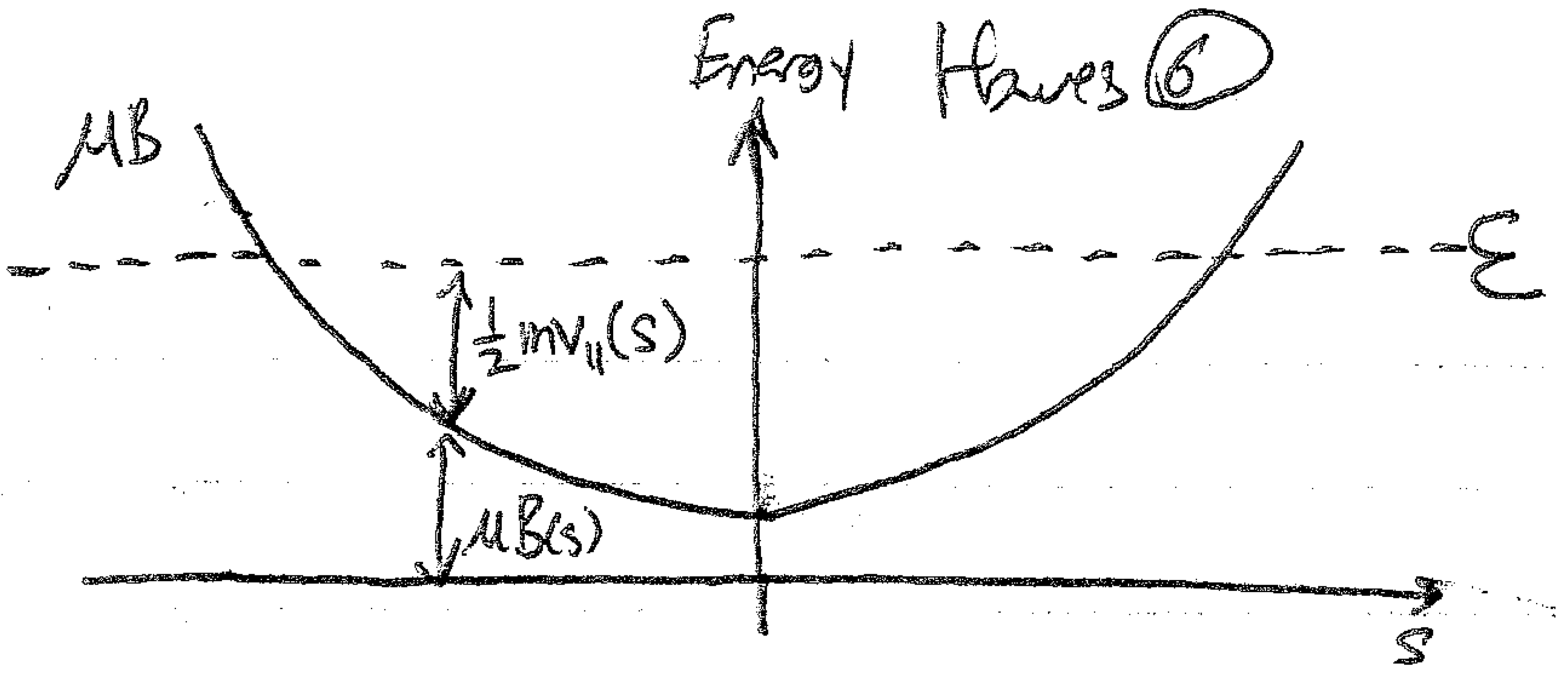
Therefore $$\boxed{\mathcal{E} = \frac{1}{2}m\, v_\parallel^2(s) + \mu\, B(s)}$$

$\mathcal{E} =$ const Conservation of Energy

$\mu =$ const Adiabatic Invariant

Lecture #6 (Continued)

IV. A. (Continued)



4. Potential Interpretation:

a. A charged particle in an electrostatic field $\underline{E} = -\nabla\phi$ has conserved energy $\mathcal{E} = \frac{1}{2}mv^2 + q\phi$

b. Here conservation involves parallel velocity and magnetic field magnitude $\mathcal{E} = \frac{1}{2}mv_\parallel^2 + \mu B$

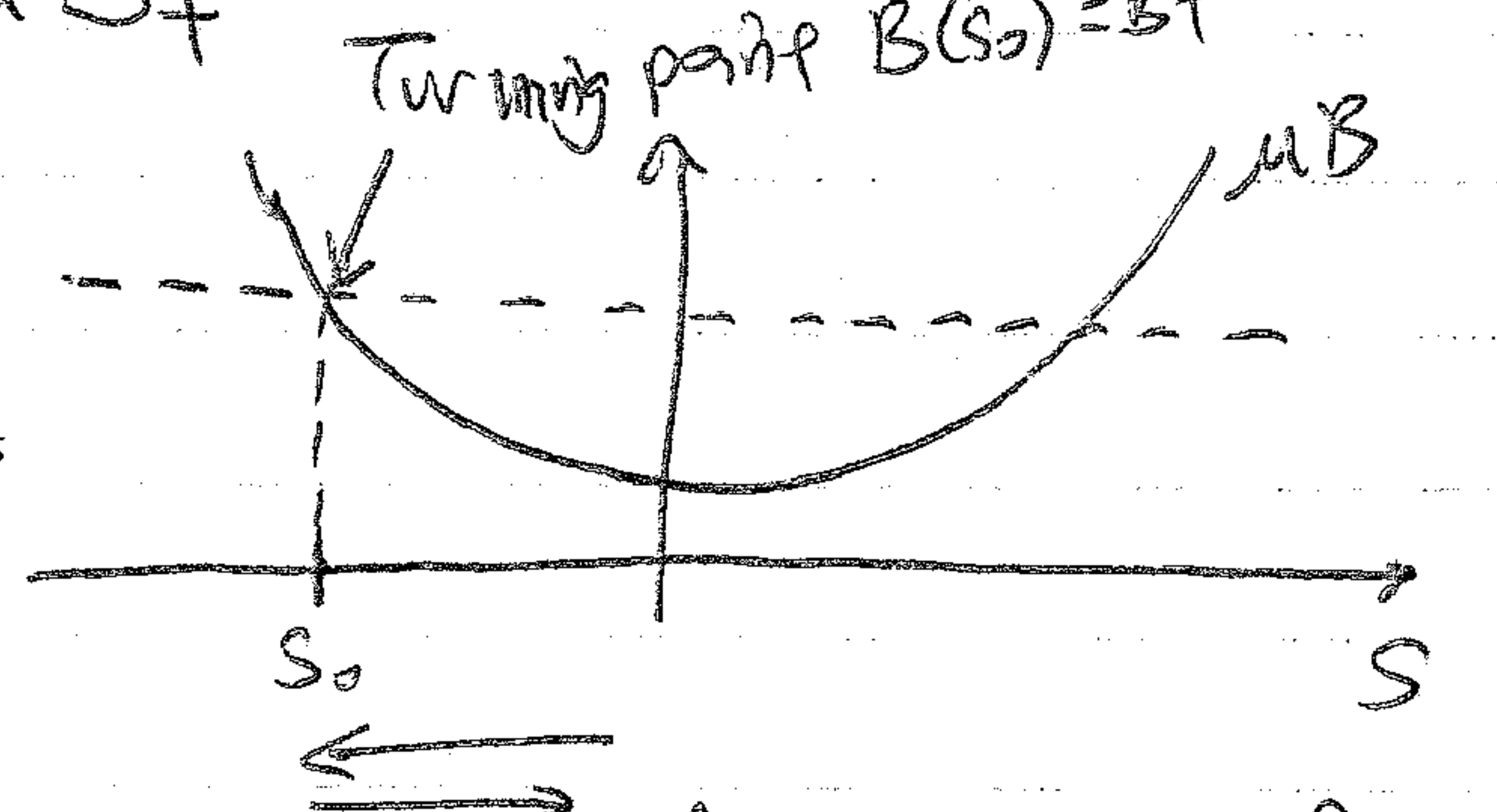
5. We can solve for $v_\parallel(s)$

$$v_\parallel(s) = \pm\sqrt{\frac{2}{m}[\mathcal{E} - \mu B(s)]}$$ where $\mathcal{E}$ and μ are constants.

a. When $v_\parallel(s_0) = 0$, the particle reaches a turning point.

Here $B(s_0) = \frac{\mu}{\mathcal{E}} \equiv B_t$

Thus $\mathcal{E} = \frac{1}{2}mv_\parallel^2 + \mu B = \mu B_t$



Particle is reflected by mirror force.

6. Physical Interpretation:

a. The particle experiences a changing $|\underline{B}|$ as it moves along field.

b. Induced azimuthal force F_ϕ does work on particle, increasing $v_\perp$

c. Total energy $\frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 = \mathcal{E}$ is conserved, so $v_\parallel$ must decrease

d. Even $v_\parallel = 0$, so the particle turns around, having been "mirrored"

7. How does pitch angle $\alpha(s)$ change? $\mathcal{E} = \frac{1}{2}mv_\parallel^2 + \mu B$ and $\mathcal{E} = \frac{1}{2}mv^2$

a. $v_\parallel = v\cos\alpha$, so $\mathcal{E} = \frac{1}{2}mv^2\cos^2\alpha + \mu B = \mathcal{E}\cos^2\alpha + \mu B$.

b. Thus $1 - \cos^2\alpha = \frac{\mu B}{\mathcal{E}}$, or $\sin^2\alpha = \frac{\mu B}{\mathcal{E}} = \frac{\mu B}{\mu B_t} = \frac{B}{B_t}$

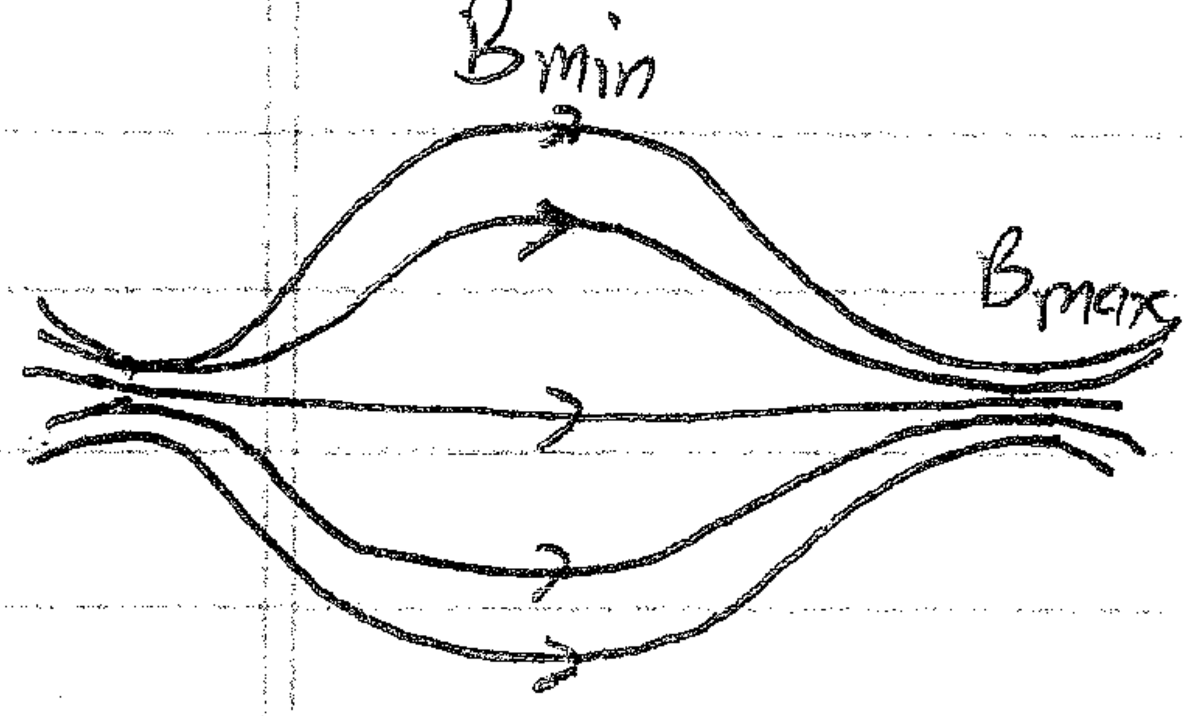
IV A. 7 (Continued)

c. Thus $\sin^2\alpha(s) = \dfrac{B(s)}{B_+}$

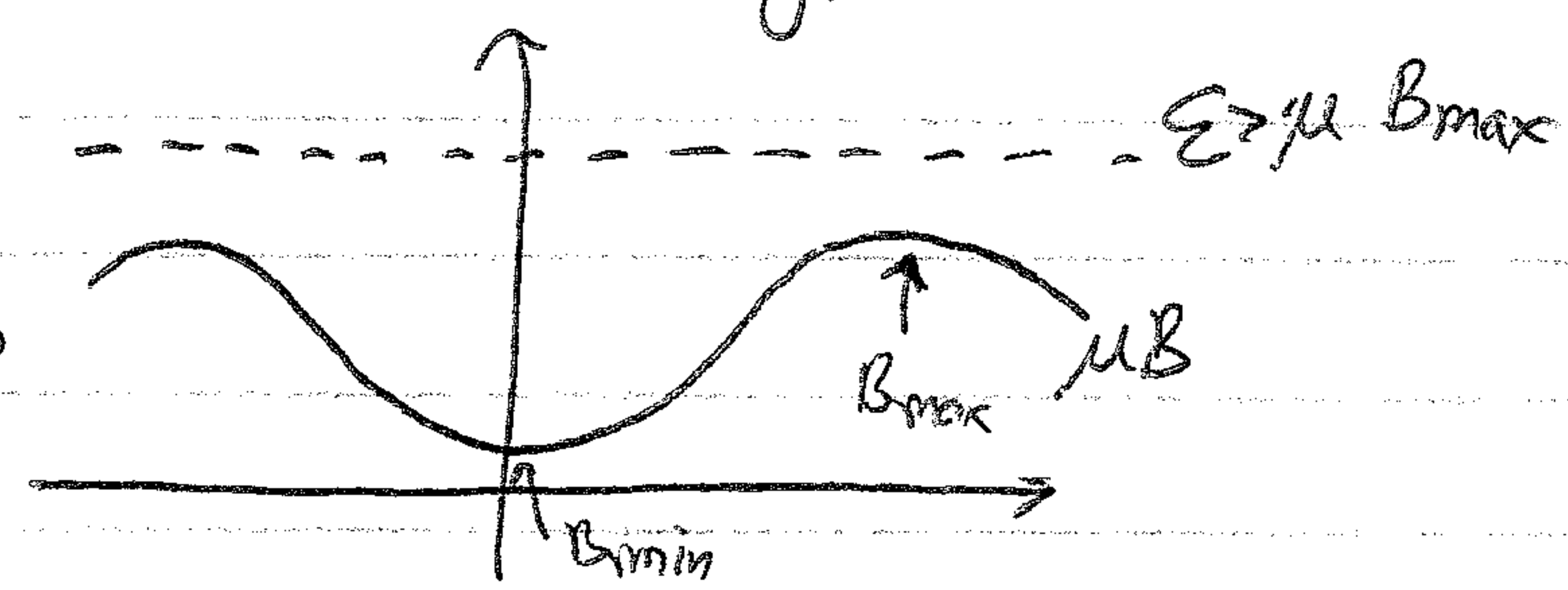
– As $B(s)$ increases to B_+, $\sin^2\alpha(s) \to 1$, or $\alpha(s) \to \frac{\pi}{2}$.

8. Practical Considerations:

a. There is a limit to the maximum field strength:



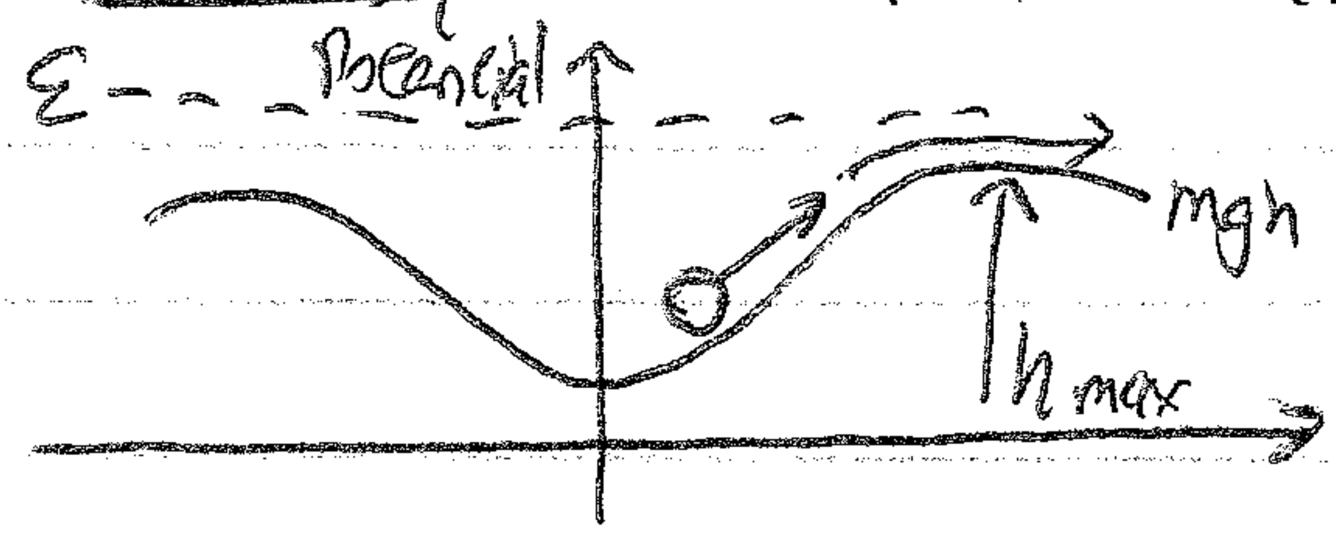
⇒



b. For Particles with $\mathcal{E} > \mu B_{max}$ $\quad v_{\parallel} = \sqrt{\frac{2}{m}(\mathcal{E} - \mu B_{max})} > 0$

Thus, $v_{\parallel}$ never reaches zero ⇒ particles are not reflected.

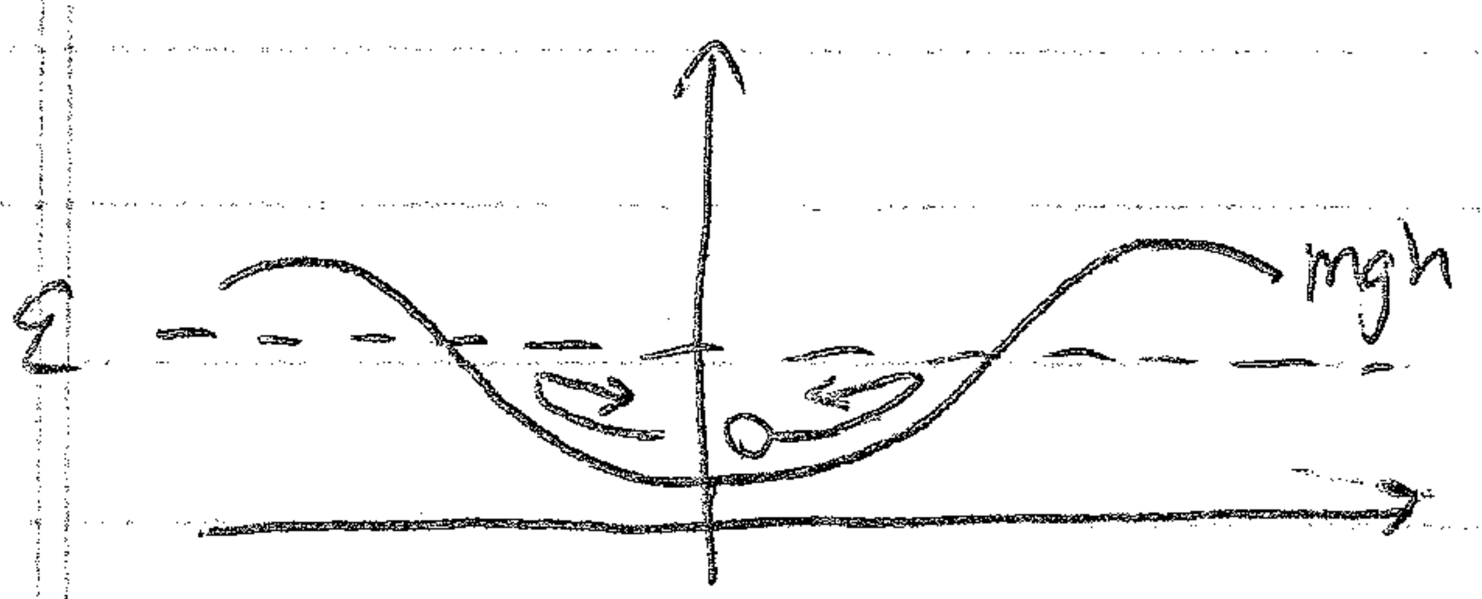
c. Analogy: Frictionless ball on a hill/valley.



$\mathcal{E} = \frac{1}{2}mv^2 + mgh$

i) If $\mathcal{E} > mgh_{max}$, ball passes over hill

ii) If $\mathcal{E} < mgh_{max}$, ball is trapped in valley, oscillating back & forth.



d. 1. We know pitch angle α increases as B increases. $\sin^2\alpha(s) = \dfrac{B(s)}{B_+}$

2. Thus, at $B = B_{min}$, pitch angle is at a minimum.

3. For a particle which reaches $\alpha = \frac{\pi}{2}$ ($v_{\parallel} = 0$) at $B = B_{max}$, what is it's pitch angle at B_0? $\Rightarrow \sin^2\alpha(s) = \dfrac{B(s)}{B_{max}}$

For $B(s) = B_{min}$,

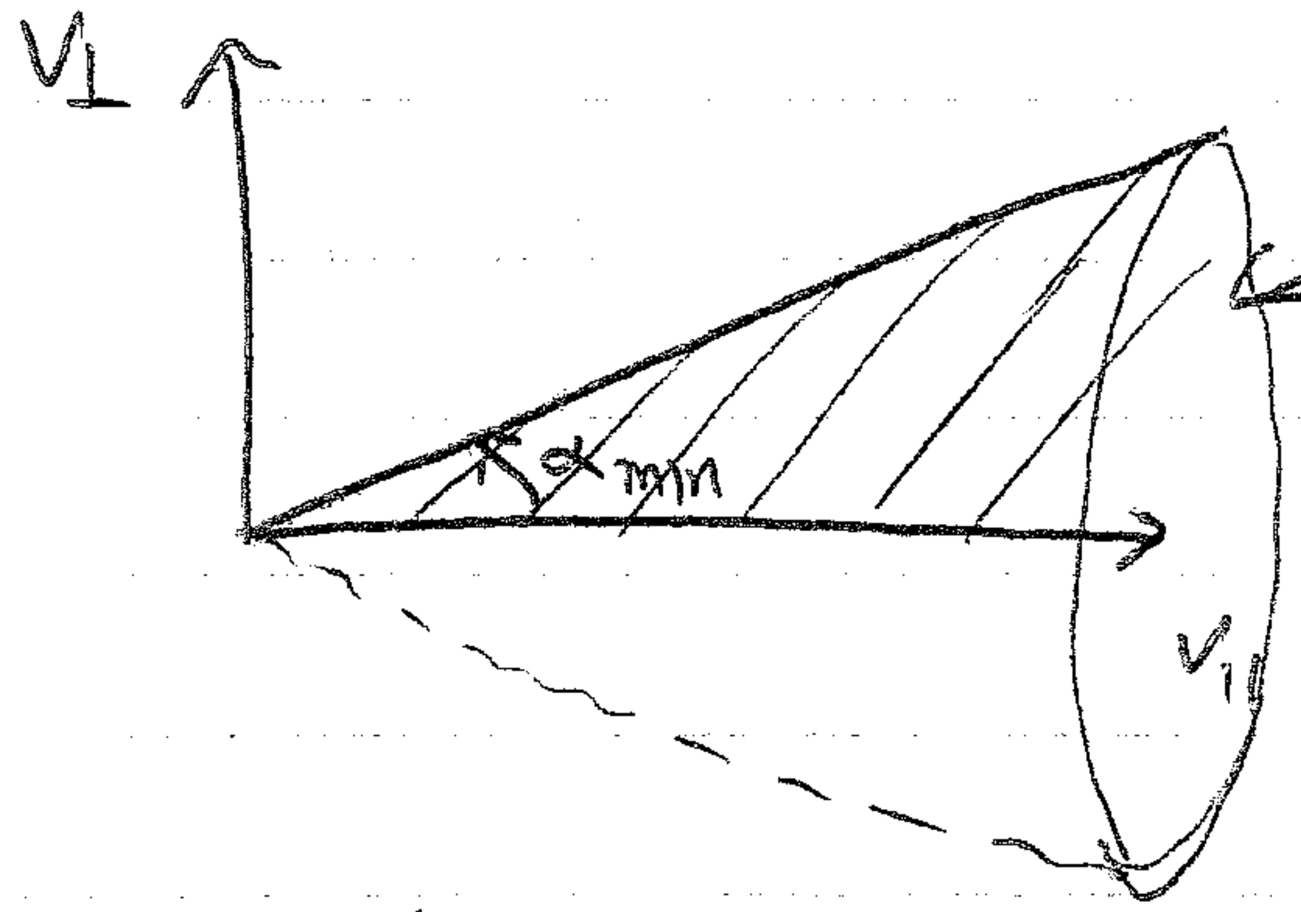
$$\boxed{\sin^2\alpha_{min} = \frac{B_{min}}{B_{max}}}$$

IV. A. 8. (Continued)

e. $\boxed{\text{Thus, for particles with } \alpha < \alpha_{min}\text{, particles will escape from magnetic mirror.}}$

f. The <u>Mirror Ratio</u> $\boxed{R_m \equiv \frac{B_{max}}{B_{min}}}$. Thus $\boxed{\sin^2\alpha_{min} = \frac{1}{R_m}}$

g. Looking in velocity space



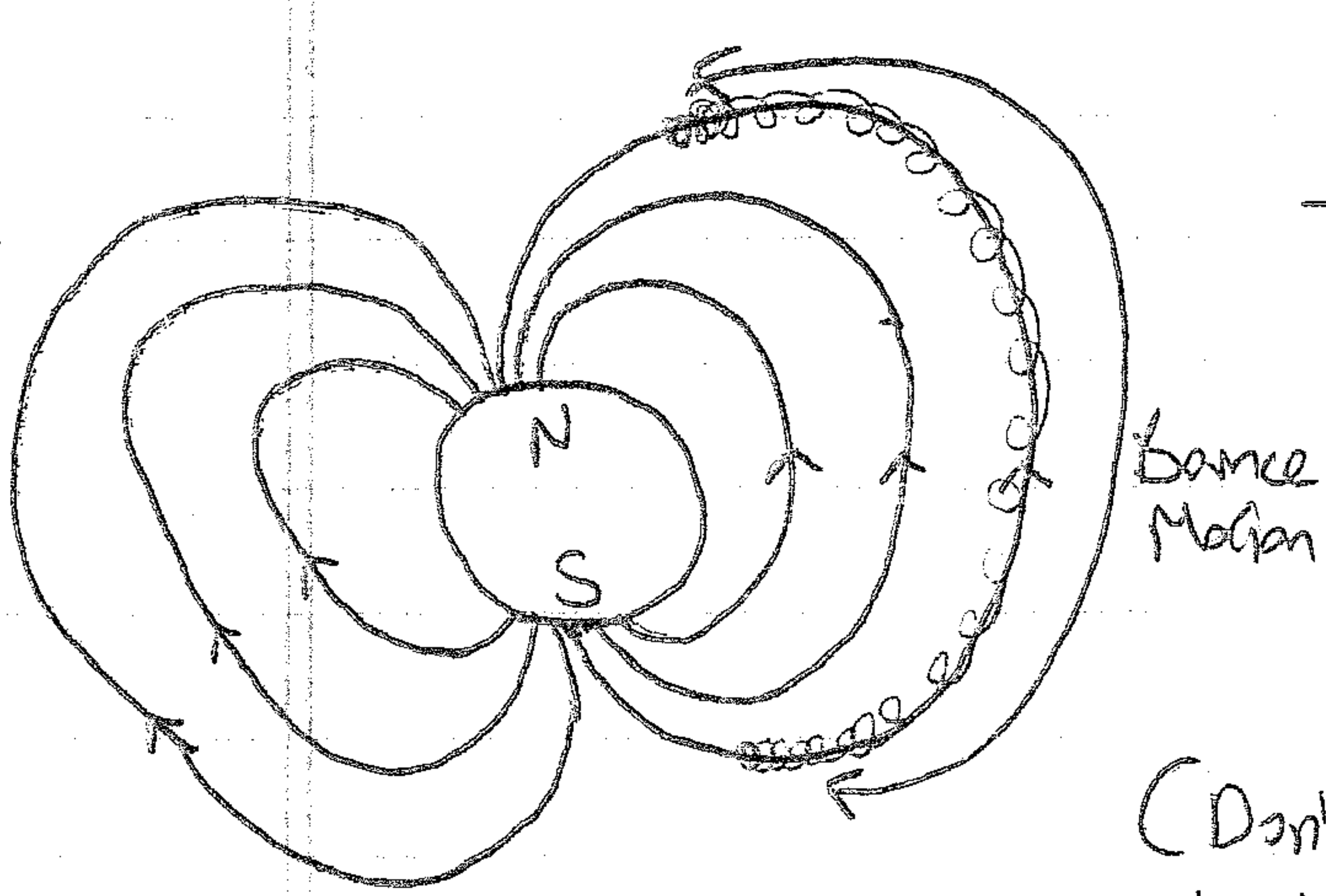
Particles with $\alpha < \alpha_{min}$ will be lost.

$\Rightarrow$ $\boxed{\text{Loss Cone}}$

h. 1. In a collisionless plasma, all particles with $\alpha < \alpha_{min}$ will be lost from mirror.

2. In a collisionless plasma, particle collisions will scatter particles into the loss cone, and eventually much of the plasma will be lost.

B. Earth's Magnetosphere:



1. Dipole field of earth behaves as a magnetic mirror
- Weak field at equator
- Strong field at poles

2. Particles trapped on field lines will bounce from pole to pole.
(Don't forget ∇B & curvature drifts also lead to motion westward around the earth)