Lecture #7: Particle Motion in Temporally Varying $\underset{\sim}{B}$ Fields & Adiabatic Invariance

I. Particle Motion in a Temporally Varying Magnetic Field $\underset{\sim}{B}(t)$

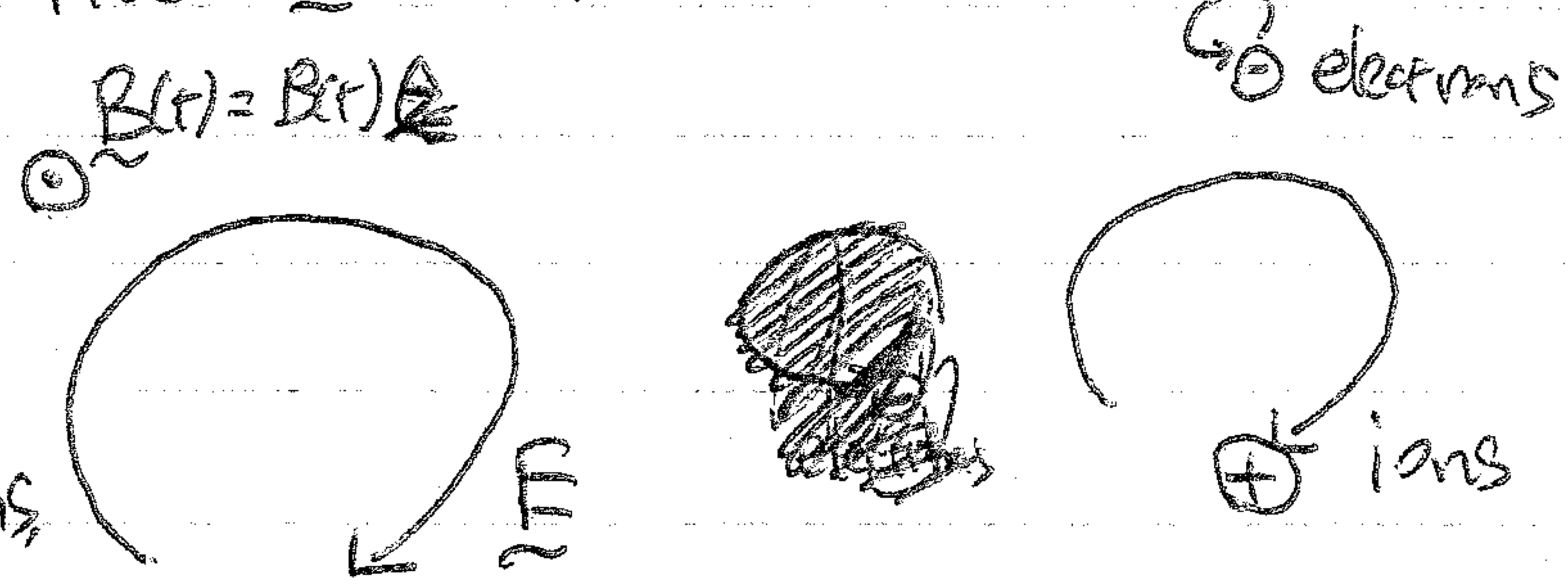
A. Uniform Magnetic Field changing in time $\underset{\sim}{B}(t) = B(t)\,\hat{\underset{\sim}{z}}$.

1. Unlike the static case, Faraday's Law tells us

$$\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E}$$, so an electric field is produced.

2. What will this Electric Field $\underset{\sim}{E}$ do?

a. Take $\frac{dB}{dt} > 0$



b. We expect the electric field can accelerate ions or electrons.

c. For $\frac{dB}{dt} > 0$, ions are accelerated in $-\hat{\phi}$ direction, electrons are accelerated in $+\hat{\phi}$ direction $\Big\}$ Both gain energy.

3. Lorentz Force Law: $m\frac{d\underset{\sim}{v}}{dt} = q\left(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B}\right)$

a. Take $\underset{\sim}{v}\cdot$ $\Rightarrow$ $m\,\underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt} = q\,\underset{\sim}{v}\cdot\left(\underset{\sim}{E} + \cancel{\underset{\sim}{v}\times\underset{\sim}{B}}^{\,0}\right)$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\,\underset{\sim}{v}\cdot\underset{\sim}{E}$$

b. For $\frac{d\underset{\sim}{B}}{dt} = \frac{dB}{dt}\hat{\underset{\sim}{z}} = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\underset{\sim}{z}}$ $\Rightarrow E_z = 0$

c. Therefore $\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\left(\frac{1}{2}m v_\perp^2\right)$ since $\frac{dv_\parallel}{dt} = 0$.

4. What is the energy change due to $\frac{d\underset{\sim}{B}}{dt} \neq 0$?

a. $$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\,\underset{\sim}{v}\cdot\underset{\sim}{E}$$

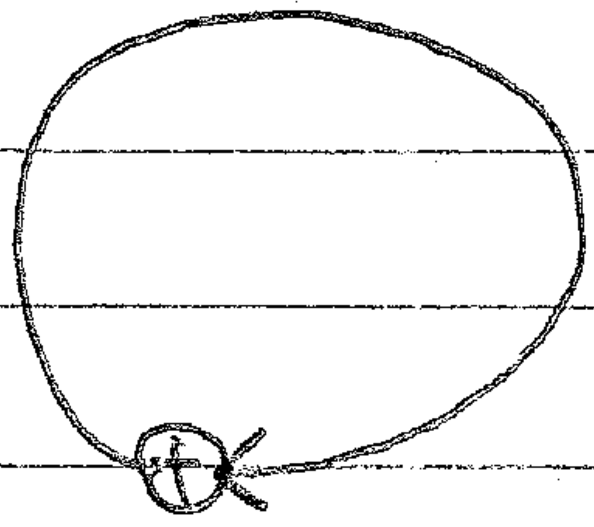
b. If $\frac{dB}{dt}$ changes slowly, we can calculate this energy change along the unperturbed Larmor orbit, $\oint_0^{2\pi/\omega_c}$

$$\oint_0^{2\pi/\omega_c} \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right)dt = q\oint_0^{2\pi/\omega_c} \underset{\sim}{E}\cdot\underset{\sim}{v}\,dt = q\oint_0^{2\pi/\omega_c} \underset{\sim}{E}\cdot\frac{d\underset{\sim}{\ell}}{dt}dt$$

Lecture #7 (Continued)

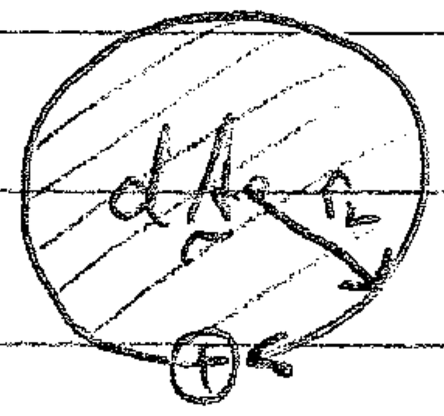
I.A.4. (Continued)

c. $$\underbrace{\Delta\left(\tfrac{1}{2}mv_\perp^2\right)}_{\text{Change over 1 orbit}} = q\oint_0^{\frac{2\pi}{\omega_c}} \underset{\sim}{E}\cdot\frac{d\underset{\sim}{\ell}}{dt}\,dt = q\oint_C \underset{\sim}{E}\cdot d\underset{\sim}{\ell}$$



Line integral over the path of the particle

d. By Stoke's Theorem, $$q\oint_C \underset{\sim}{E}\cdot d\underset{\sim}{\ell} = q\oint_S \nabla\times\underset{\sim}{E}\cdot d\underset{\sim}{A} = -q\oint_S \frac{d\underset{\sim}{B}}{dt}\cdot d\underset{\sim}{A}$$



Surface integral over area enclosed by the Larmor orbit

e. NOTE: For the ion motion above, $d\underset{\sim}{A} = -dA\,\hat{\underset{\sim}{z}}$ (right-hand rule), so

$$\Delta\left(\tfrac{1}{2}mv_\perp^2\right) = -q\oint_S \frac{dB}{dt}\hat{\underset{\sim}{z}}\cdot\left(-dA\,\hat{\underset{\sim}{z}}\right) = +q\oint \frac{dB}{dt}\,dA = q\frac{dB}{dt}\left(\pi r_L^2\right)$$

f. If we assume the rate of energy change is approximately constant over Larmor orbit, $\Delta\left(\tfrac{1}{2}mv_\perp^2\right) = \frac{dW_\perp}{dt}\Delta t = \frac{dW_\perp}{dt}\left(\frac{2\pi}{\omega_c}\right)$ where $W_\perp = \tfrac{1}{2}mv_\perp^2$ is perpendicular energy.

g. Thus

$$\frac{dW_\perp}{dt} = \frac{\omega_c}{2\pi}q\frac{dB}{dt}\pi\underbrace{\left(\frac{v_\perp^2}{\omega_c^2}\right)}_{r_L^2=\frac{v_\perp^2}{\omega_c^2}} = \frac{q\,v_\perp^2}{2}\left(\frac{m}{qB}\right)\frac{dB}{dt} = \left(\frac{mv_\perp^2}{2B}\right)\frac{dB}{dt} = \mu\frac{dB}{dt}.$$

h. Since $\mu = \frac{W_\perp}{B}$, $$\frac{dW_\perp}{dt} = \frac{W_\perp}{B}\frac{dB}{dt} \Rightarrow \int\frac{dW_\perp}{W_\perp} = \int\frac{dB}{B}$$

$$\Rightarrow \ln W_\perp = \ln B + C \Rightarrow \frac{W_\perp}{B} = \text{constant.}$$

5. Therefore, for slowly varying magnetic fields $\underset{\sim}{B}(t)$,

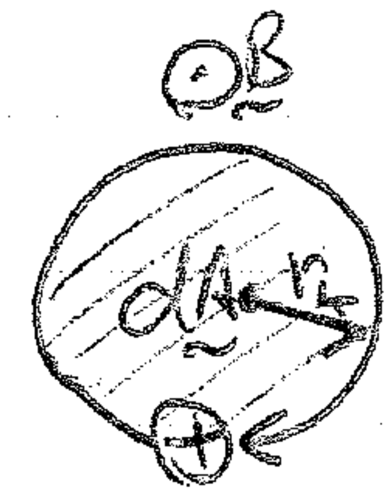
$$\boxed{\frac{d\mu}{dt} = \text{constant}}$$

I. (Continued)

B. Magnetic Flux Interpretation

1. Conservation of the Magnetic Moment μ is equivalent to maintaining a constant magnetic flux through Larmor orbits.

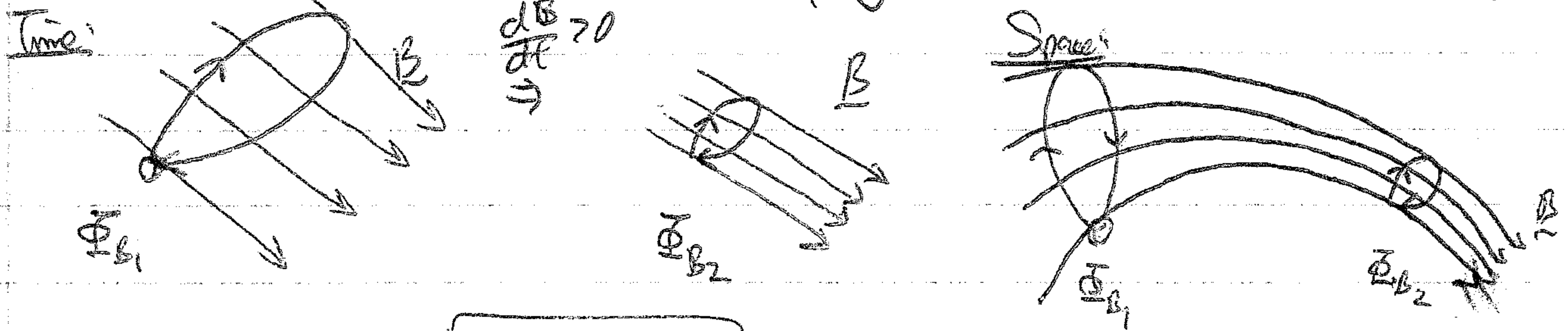
a. $\Phi_B = d\underset{\sim}{A} \cdot \underset{\sim}{B}$



For our case $\Phi_B = B\,dA = B\pi r_L^2 = \dfrac{\pi v_\perp^2}{\Omega_c^2} B = \pi \dfrac{v_\perp^2 m^2}{q^2 B^2} \cancel{B} = \dfrac{2\pi m}{q^2}\left(\dfrac{m v_\perp^2}{2B}\right)$

$$\boxed{\Phi_B = \frac{2\pi m}{q^2}\mu}$$

2. This holds for a $\underset{\sim}{B}$-field varying (slowly) in either time or space.



$$\boxed{\Phi_{B_1} = \Phi_{B_2}}$$

II. Adiabatic Invariance

A. General Result from Hamiltonian Mechanics

For "nearly" periodic system & slowly varying parameters,

Action Integral

$$J = \oint p\, dq$$

is an adiabatic invariant

i. p & q are conjugate momentum & position coordinates.

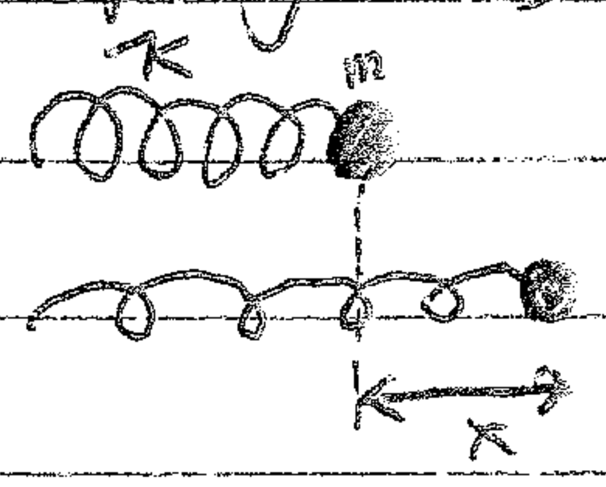
I. (Continued)

B. Example: Harmonic Oscillator

1. Consider a time dependent harmonic oscillator.

$$\frac{d^2x}{dt^2} + \omega(t)\, x = 0$$

Ex: Spring-mass system



$$\omega = \frac{k}{m}$$

2. Position $q = x = A \sin \omega t$

Momentum $p = m v_x = m\omega A \cos \omega t$

3. Action Integral:

$$J = \oint p\, dq = \oint_0^{2\pi/\omega} m\omega A \cos\omega t \; d(A \sin\omega t) = mA^2\omega^2 \oint_0^{2\pi/\omega} \cos^2\omega t\, dt$$

$$= mA^2\omega^2 \frac{\pi}{\omega} = \pi m \omega A^2$$

a. Thus $\boxed{J = \pi m \omega A^2 = \text{constant}}$ if $\omega(t)$ changes slowly.

b. So amplitude $A \propto \omega^{-\frac{1}{2}}$. If frequency decreases, amplitude will increase.

4. Total Energy $W = \frac{p^2}{2m} = \frac{1}{2} m \omega^2 A^2$, so this can also be written

$$J = 2\pi \frac{W}{\omega} = \text{constant}.$$

C. <u>How slow must system change to satisfy invariance?</u>

1. Since amplitude $A \propto \frac{1}{\omega^{\frac{1}{2}}}$, consider the WKB solution

$$X_{WKB} = \frac{1}{\sqrt{\omega(t)}}\, e^{\pm i \int^t \omega(t')\, dt'}$$

a. In this case, $J = \pi m \omega^2 A^2$ is precisely constant.

2a. This solution is an <u>exact</u> solution of the differential equation

$$\frac{d^2 X_{WKB}}{dt^2} + \left[\omega^2 + \frac{\ddot{\omega}}{2\omega} - \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2\right] X_{WKB} = 0$$

Lecture #7 (Continued)

II. C2 (Continued)

b. Here $\dot{\omega} = \frac{d\omega}{dt}$ and $\ddot{\omega} = \frac{d^2\omega}{dt^2}$.

3. The WKB solution is a good approximation when

$$\omega^2 \gg \left|\frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 - \frac{\ddot{\omega}}{2\omega}\right|$$

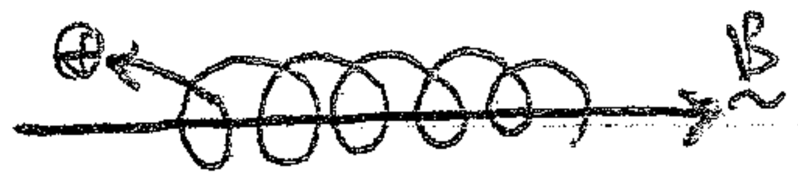
4. Rule of thumb:

> The adiabatic invariant is approximately constant when the change of characteristic frequency is small over one period.

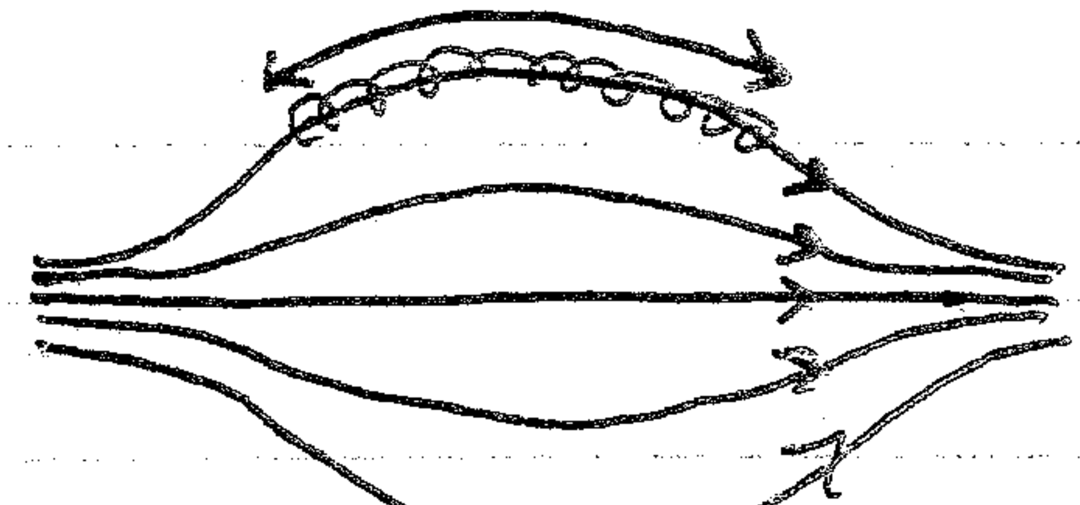
D. Example: Magnetic Mirror and its first, second, and third adiabatic invariants

1. Three types of periodic motion in an axisymmetric magnetic mirror:

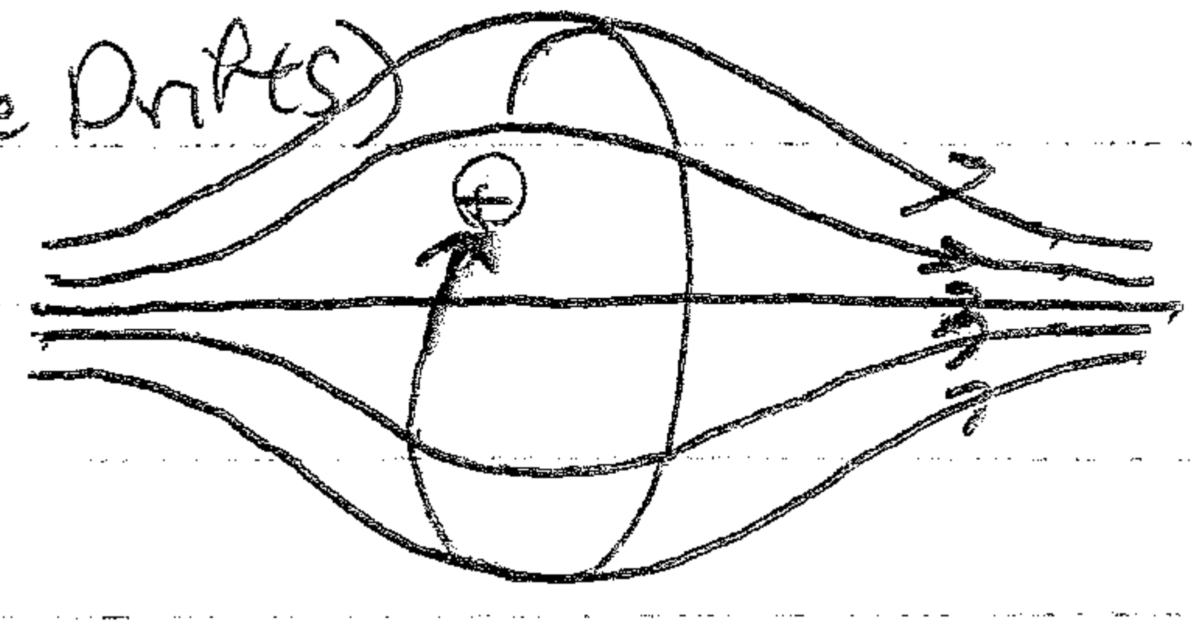
a. Larmor Motion



b. Parallel Bounce Motion



c. Azimuthal Drift Motion
(due to ∇B and Curvature Drifts)



2. First Adiabatic Invariant:

a. As we know from lecture #3, the lowest order motion in a magnetic field is Larmor motion, $\frac{d^2 v_1}{dt^2} = -\omega_c^2 v_1$ or $\frac{d^2 x}{dt^2} = -\omega_c^2 x$

II. D.2 (Continued)

b. In this case, the action integral is

$$\boxed{J_1 = \pi m \omega_c r_L^2}$$ using $x = r_L \sin \omega_c t$, $V_x = r_L \omega_c \cos \omega_c t$

c. NOTE that this can be written

$$J = \pi m \frac{qB}{m} \frac{v_\perp^2}{\left(\frac{qB}{m}\right)^2} = \frac{2\pi m}{q} \left(\frac{m v_\perp^2}{2B}\right) = \frac{2\pi m}{q} \mu$$

This is just the same as μ (with a constant factor)

3. Second Adiabatic Invariant, (Parallel Bounce Motion)

a. The action integral for parallel bounce motion

$$J = m \oint v_\parallel ds$$ $s \equiv$ distance along magnetic field.

b. We know, for a turning point at $B = B_t$,

$$\frac{1}{2} m v_\parallel^2 + \mu B(s) = \mu B_t$$ (Lecture #6)

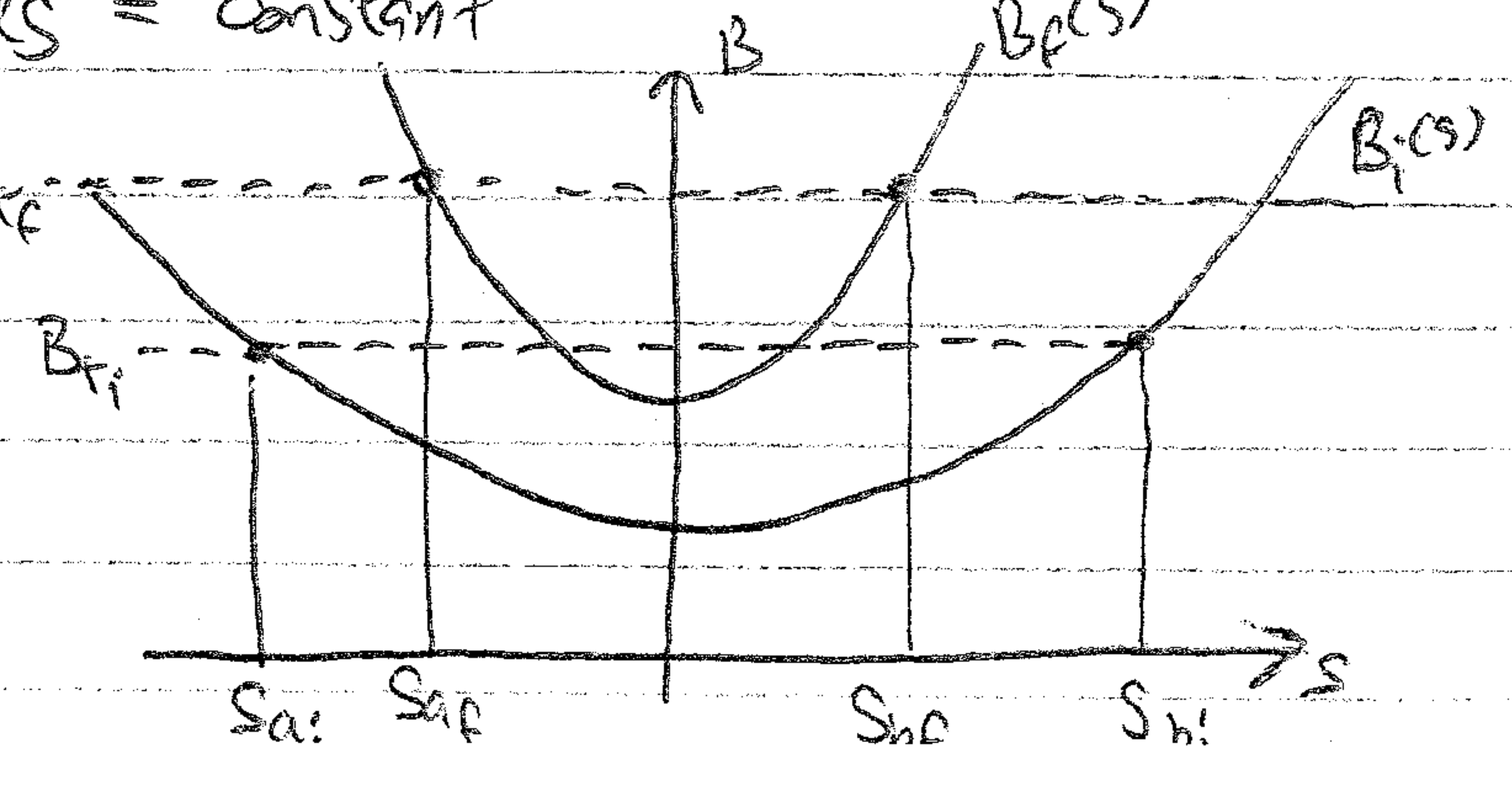
So $$v_\parallel(s) = \pm \sqrt{\frac{2\mu}{m}} \sqrt{B_t - B(s)}$$

c. This gives $$\boxed{J_2 = \sqrt{2\mu m} \oint \sqrt{B_t - B(s)}\, ds}$$

d. Thus, for a given magnetic field configuration with $B(s)$,

$$\int_{s_a}^{s_b} \sqrt{B_t - B(s)}\, ds = \text{constant}$$

for bounce motion between two points s_a & s_b



II. D3 (Continued)

e. As illustrated above, the constancy of J_2 (for slowly varying system parameters) can be used to determine new motion of a system.

i. For an initial magnetic field $B_i(s)$ and initial energy, we may calculate J_2 and s_{ai} & s_{bi}.

ii. Let the magnetic field change (slowly) from $B_i(s)$ to $B_f(s)$.

iii. Since $J_{2i} = \int_{s_{ai}}^{s_{bi}} \sqrt{B_{+i} - B(s)}\, ds = \int_{s_{af}}^{s_{bf}} \sqrt{B_{tf} - B(s)}\, ds$,

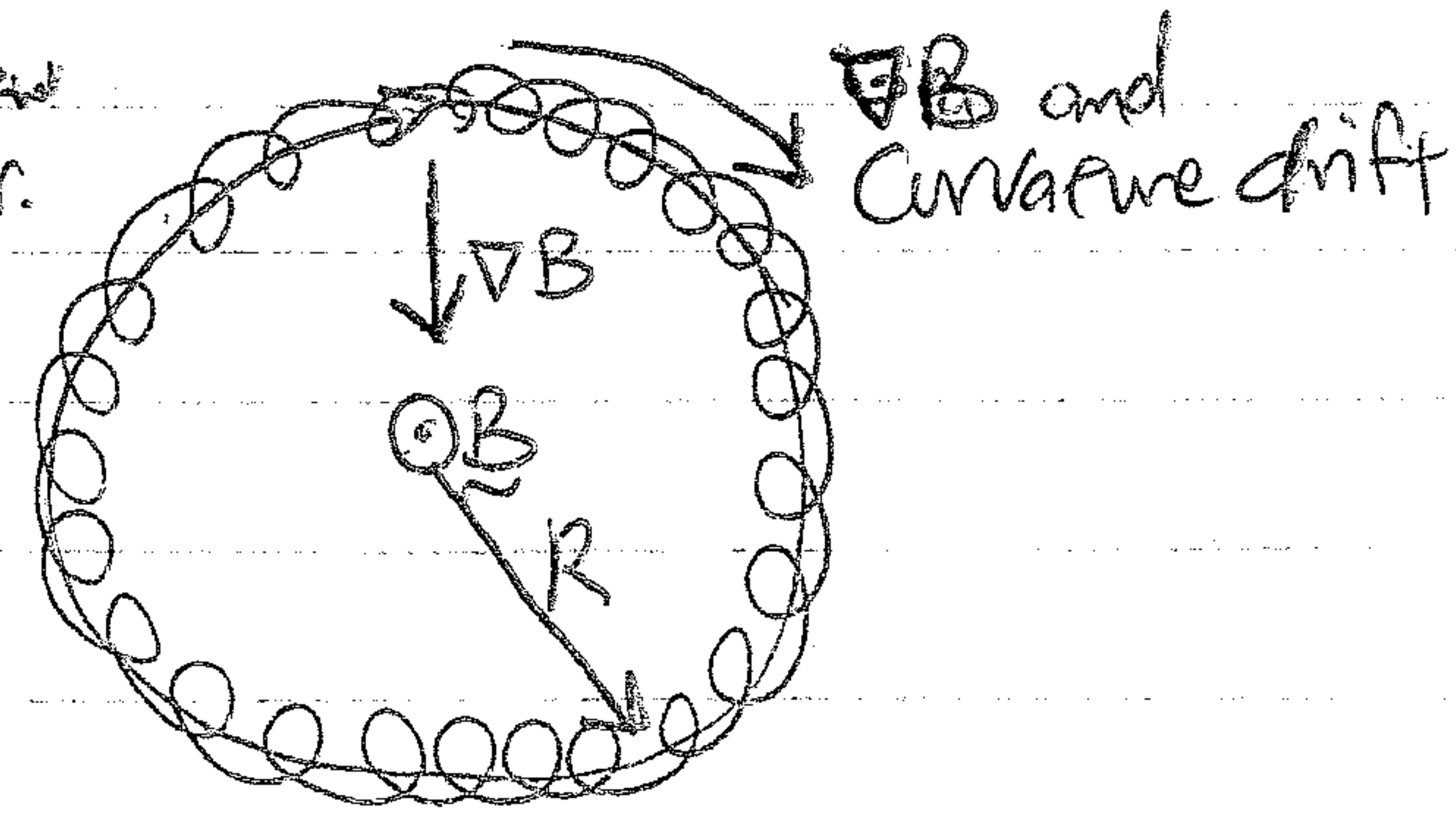
we can adjust B_{tf} (and find corresponding mirror points s_{af} & s_{bf}) ~~and~~ until this integral is satisfied using $B_f(s)$.

iv. The final total energy is then μB_{tf} (since μ = const).

4. Third Adiabatic Invariant, (Azimuthal Drift Motion)

a. This invariant only exists in axially symmetric cases, such that the drift orbits of the guiding centers are nearly closed.

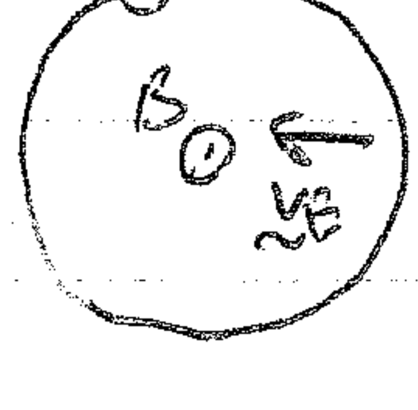
End view of Mirror.



b. What happens when $\underline{B}(t)$ changes in time?

i. $\int_C \underline{E} \cdot d\underline{\ell} = -\int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{A}$ is change in energy

ii. Assuming Axisymmetry, $E(2\pi R) = -\frac{dB}{dt} \pi R^2 \Rightarrow E = -\frac{R}{2}\frac{dB}{dt}$

iii. 

$\underline{E} = -\frac{R}{2}\frac{dB}{dt}\hat{\underline{\phi}}$ Produce $\underline{E}\times\underline{B}$ drift radially inward.

II. D. 4. b. (Continued)

iv. $\underset{\sim}{V}_E = \frac{\underset{\sim}{E} \times B_z \hat{z}}{B_z^2} = \frac{-\frac{R}{2}\frac{dB}{dt}\hat{\phi} \times B_z \hat{z}}{B_z^2} = -\frac{R}{2B}\frac{dB}{dt}\hat{\underset{\sim}{r}}$

But $v_E = \frac{dR}{dt} \Rightarrow \frac{dR}{dt} = \frac{-R\, dB}{2B\, dt} \Rightarrow \frac{2\, dR}{R} = -\frac{dB}{B}$

v. Thus $R^2 B = \text{const.}$

vi. The Magnetic Flux through drift orbit is

$$\Phi_B = \pi R^2 B = \text{const.}$$

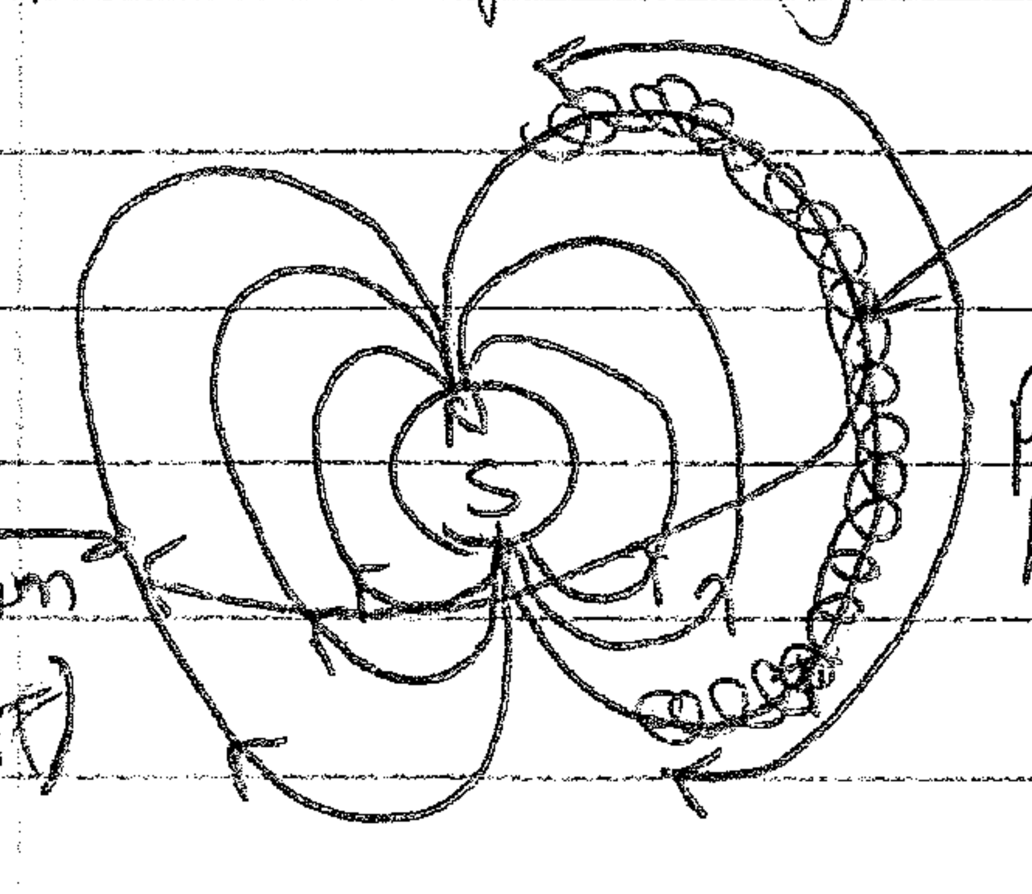
(Assuming B is relatively constant near axis of symmetry.)

vii. Thus, the 3rd Adiabatic Invariant means the

$\boxed{\text{Flux enclosed by drift orbit remains constant}}$

Particle remains on the surface of a flux tube.

E. Example: Magnetosphere.

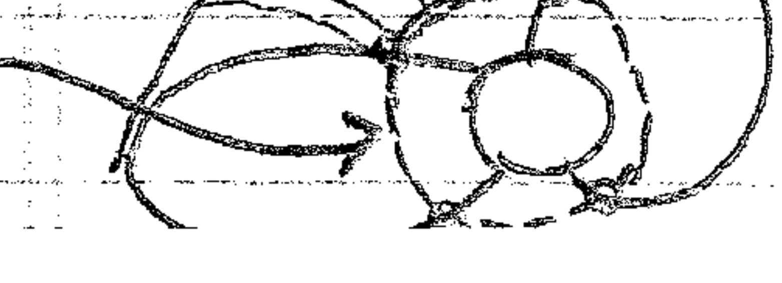


1. Second Adiabatic Invariant applies even without axisymmetry.
2. a. For a constant Magnetic field ($\frac{dB}{dt}=0$), energy is conserved.

$\underset{\text{const}}{\mathcal{E}} = \frac{1}{2}mv_{\parallel}^2 + \underset{\text{const}}{\mu} B(s)$ b. Since $\mathcal{E} = \text{const}$, $B_+ = \text{constant}$.

c. Factoring out B_+, $I = \int_a^b \sqrt{1 - \frac{B(s)}{B_+}}\, ds = \text{const.}$

3. Higher order multiple moments:



a. Constant B_+ & I mean that drifting particles remain on a surface $\Rightarrow$ $\boxed{\text{L-shell}}$