Lecture 7: Particle Motion in Temporally Varying \( B(t) \) Fields & Adiabatic Invariance

I. Particle Motion in a Temporally Varying Magnetic Field \( B(t) \)

A. Uniform Magnetic Field changing in time \( \frac{dB}{dt} = B(t) \).

1. Unlike the static case, Faraday's Law tells us
   \[ \frac{\partial B}{\partial t} = -\nabla \times E \], so an electric field is produced.

2. What will this Electric Field \( E \) do?
   a. Take \( \frac{dB}{dt} > 0 \)

   b. We expect the electric field can accelerate ions or electrons

   c. For \( \frac{dB}{dt} > 0 \), ions are accelerated in \(-\phi\) direction, electrons are accelerated in \(+\phi\) direction

3. Lorentz Force Law: \( m \frac{d\mathbf{v}}{dt} = q \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \)
   a. Take \( \mathbf{v} = 0 \) \( \rightarrow m \frac{d\mathbf{v}}{dt} = q \mathbf{v} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \)

   \[ \frac{d}{dt} \left( \frac{1}{2} m \mathbf{v}^2 \right) = q \mathbf{v} \cdot \mathbf{E} \]

   b. For \( \frac{dB}{dt} = \frac{d^2 B}{dt^2} = \left( \frac{\partial \mathbf{E}}{\partial \mathbf{r}} - \frac{\partial \mathbf{E}}{\partial \mathbf{y}} \right) \mathbf{r} \Rightarrow E_z = 0 \)

   c. Therefore \( \frac{d}{dt} \left( \frac{1}{2} m \mathbf{v}^2 \right) = \frac{d}{dt} \left( \frac{1}{2} m \mathbf{v}^2 \right) \) since \( \frac{d\mathbf{v}}{dt} = 0 \).

4. What is the energy change due to \( \frac{dB}{dt} \neq 0 \)?
   a. \[ \frac{d}{dt} \left( \frac{1}{2} m \mathbf{v}^2 \right) = q \mathbf{v} \cdot \mathbf{E} \]

b. If \( \frac{dB}{dt} \) changes slowly, we can calculate this energy change along the unperturbed Larmor orbit \( \phi_0 \)

\[ \frac{d}{dt} \left( \frac{1}{2} m \mathbf{v}^2 \right) dt = q \phi_0 \mathbf{E} \cdot d\mathbf{r} = q \phi_0 \mathbf{E} \cdot \mathbf{r} \]
c. \[ \Delta (\frac{1}{2}mv^2) = q \oint_C E \cdot dl = q \oint_C E \cdot dl \]

Change over 1 orbit

Line integral over the path of the particle

By Stokes' Theorem, \[ q \oint_C E \cdot dl = q \oint_S \nabla \times E \cdot dA = -q \oint_S \frac{dB}{dt} \cdot dA \]

Surface integral over area enclosed by the Larmor orbit

Note: For the ion motion above, \( dA = -dA \frac{e}{m} \) (right-hand rule), so

\[ \Delta (\frac{1}{2}mv^2) = -q \oint_S \frac{dB}{dt} \cdot dA = -q \oint_S \frac{dB}{dt} \cdot dA = q \oint_S \frac{dB}{dt} \]

If we assume the rate of energy change is approximately constant over Larmor orbit, \( \Delta (\frac{1}{2}mv^2) = \frac{dW}{dt} \Delta t = \frac{dW}{dt} \frac{2\pi}{v_c} \)

Where \( W = \frac{1}{2}mv^2 \) is perpendicular energy,

Thus

\[ \frac{dW}{dt} = \frac{ev}{2 \pi} \frac{dB}{dt} \left( \frac{u^2}{v_c^2} \right) = \frac{q}{2 \pi} \frac{m}{v_c} \frac{dB}{dt} = \left( \frac{mv^2}{2B} \right) \frac{dB}{dt} = B \frac{dB}{dt} \]

Since \( N = \frac{W_1}{B} \),

\[ \frac{dW}{dt} = \frac{W_1}{B} \frac{dB}{dt} \Rightarrow \left( \frac{dW}{dt} \right) B = \frac{W_1}{B} \frac{dB}{dt} \]

\[ \Rightarrow \frac{dW}{dt} = \mu \frac{dB}{dt} \Rightarrow \frac{dW}{dt} = \mu \frac{dB}{dt} = \text{constant} \]

Therefore, for slowly varying magnetic fields \( B(t) \),

\[ \frac{dW}{dt} = \text{constant} \]
B. Magnetic Flux Interpretation

1. Conservation of the Magnetic Moment $\mathbf{A}$ is equivalent to maintaining a constant magnetic flux through Larmor orbit.

\[ \Phi_B = \oint \mathbf{A} \cdot d\mathbf{r} \]

For our case:

\[ \Phi_B = B \pi n^2 = \frac{\pi}{2} \frac{v^2}{c^2} \rho^2 = \pi \frac{v^2}{c^2} \frac{m^2}{q^2} \tau^2 = \frac{2 \pi m^2}{q^2} (\frac{m v^2}{2 \hbar}) \]

\[ \Phi_B = \frac{2 \pi m}{q^2} \mu \]

2. This holds for a $B$-field varying (slowly) in either time or space.

II. Adiabatic Invariance

A. General Result From Hamiltonian Mechanics

For "nearly" periodic system & slowly varying parameters, the action integral

\[ J = \oint p \, dq \]

is an adiabatic invariant.

1. $p$ & $q$ are conjugate momentum & position coordinates.
II. (Continued)

8. Example: Harmonic Oscillator

1. Consider a time dependent harmonic oscillator.

\[ \frac{d^2 x}{dt^2} + \omega(t) x = 0 \]

- E: Spring-mass system

\[ \omega = \frac{K}{m} \]

2. Position \( q = x = A \sin \omega t \)

- Momentum \( p = mv = ma \Omega \cos \omega t \)

3. Action Integral:

\[ J = \oint pdq = \oint_0^{2\pi} m v A \cos \omega t \, d(\sin \omega t) = mA^2 \int_0^{2\pi} \cos^2 \omega t \, d\omega \]

\[ = \frac{mA^2 \omega}{2} \int_0^{2\pi} \cos \omega t \, d\omega = \pi m A^2 \omega \]

- This integral \( J = \pi mA^2 \) is constant if \( \omega(t) \) changes slowly.

- So amplitude \( A \propto \omega^{-1/2} \). If frequency decreases, amplitude will increase.

4. Total Energy \( W = \frac{p^2}{2m} = \frac{1}{2} m A^2 \omega^2 \), so this can also be written

\[ J = 2\pi W \frac{\omega}{\omega} = \text{constant} \]

C. How slow must system change to satisfy invariance?

1. Since amplitude \( A \propto \frac{1}{\omega} \), consider the WKB solution

\[ X_{WKB} = \frac{1}{\sqrt{2\pi}} e^{-i \int \omega(t) \, dt} \]

- In this case, \( J = \pi mA^2 \) is precisely constant.

2. This solution is an exact solution of the differential equation

\[ \frac{d^2 X_{WKB}}{dt^2} + \left[ \omega^2 + \frac{\dot{\omega}^2}{2\omega} - \frac{3}{4} (\frac{\dot{\omega}}{\omega})^2 \right] X_{WKB} = 0 \]
Lecture 47 (Continued)

II. Continued

11. Here \( \dot{\omega} = \frac{d\omega}{df} \) and \( \ddot{\omega} = \frac{d^2\omega}{df^2} \).

3. The WKB solution is a good approximation when

\[
\omega^2 \gg \left| \frac{3\dot{\omega}}{4} \left( \frac{\dot{\omega}}{\omega} \right)^2 - \frac{\ddot{\omega}}{2\omega} \right|
\]

Rule of Thumb

The adiabatic invariant is approximately constant when the change of characteristic frequency is small over one period.

IV. Example: Magnetic Mirror and its First, Second, and Third Invariants

1. Three types of periodic motion in an axisymmetric magnetic mirror:
   a. Larmor Motion
   b. Parallel Bounce Motion
   c. Azimuthal Drift Motion (due to D8 and Curvature Drifts)

2. First Adiabatic Invariant:
   a. As we know from lecture #3, the lowest order motion in a magnetic field is Larmor motion:

\[
\frac{d^2 \mathbf{V}}{dt^2} = -e \mathbf{V} \times \mathbf{B} \quad \text{or} \quad \frac{d^2 \mathbf{x}}{dt^2} = -e \mathbf{E}
\]
b. In this case, the action integral is

\[ J = \int \alpha \, \text{max} \, v_z^2 \, \text{d}t \]

using \( x = r \, \sin \theta \),
\( v_x = r \, \cos \theta \),
\( v_x = r_c \, \cos \theta \) \text{ constant}.

\[ J = \frac{9}{4} \, m^2 \, g_\| \, \frac{v_z^2}{(g_\| / m)^2} = \frac{2}{5} \, m \, v_z^2 \left( \frac{m}{2 \, \beta} \right) = \frac{2}{5} \, m \, v_z^2 \] \text{This is just the same as} \, \mu \text{ (with a constant factor).}

3. Second Adiabatic Invariant (Source Motion)
   a. The action integral for parallel bounce motion

\[ J = m \int \, 0 \, v_{\parallel} \, \text{d}s \quad \text{s = distance along magnetic field.} \]

b. We know, for a turning point at \( B = B_t \),
\[ \frac{1}{2} \, m \, v_{\parallel}^2 + \mu \, B = \mu \, B_t \] (Leaves #6)

So
\[ v_{\parallel}(s) = \pm \sqrt{2 \mu \, m} \sqrt{B_t - B(s)} \]

c. This gives
\[ J = \int \sqrt{1/2 \mu \, m} \, \sqrt{B_t - B(s)} \, \text{d}s \]

d. Thus, for a given magnetic field configuration with \( B(s) \),
\[ \int_{s_0}^{s_f} \sqrt{B_t - B(s)} \, \text{d}s = \text{constant} \]

for bounce motion \( B_t \) between two points \( s_0 \) and \( s_f \).

\( B_t \) between two points \( s_0 \) and \( s_f \).
e. As illustrated above, the constancy of $J_2$ (for slowly varying system parameters) can be used to determine new motion of a system.

i. For an initial magnetic field $B_i(t)$ with initial energy, we may calculate $J_2$ and $S_{ai}$ & $S_{bi}$.

ii. Let the magnetic field change (slowly) from $B_i(t)$ to $B_f(t)$.

iii. Since $J_2 = \left. \int S_{bi} \sqrt{B_i - B(s)} \, ds \right|_{S_{ai}}^{S_{bf}} = \left. \int S_{bf} \sqrt{B_f - B(s)} \, ds \right|_{S_{af}}^{S_{bf}}$, we can adjust $B_f$ (and find corresponding mirror points $S_{af}$ & $S_{bf}$) and until this integral is satisfied using $B_f(t)$.

iv. The final total energy is then $\mu B_f$ (since $\mu_0 = \mu$).

4. Third Adiabatic Invariant, (Azimuthal Drift Motion)

a. This invariant only exists in initially symmetric cases, such that the drift orbits are nearly closed.

b. What happens when $B(t)$ changes in time?

i. $\int C.E. \, dt = -\int \frac{\partial E}{\partial t} \, dt$ is change in energy

ii. Assuming axial symmetry, $E(\theta, R) = -\frac{dB}{dt} \theta R^2 \Rightarrow E = -\frac{B}{2} \frac{dB}{dt}$

iii. $E = -\frac{R d^2 B}{2} \times \hat{B}$ Produce $E \times B$ drift radially inward.

End view of mirror. \( \text{Curvature drift} \)

\( \nabla B \) and curvature drift
Lecture #7 (Continued)

4. D. (b) (Continued)

iv. \[ v_e = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = -\frac{\mathbf{R} \times \mathbf{B}}{2\pi} \times \mathbf{R} \times \mathbf{B} \times \mathbf{B} = -\frac{\mathbf{R} \cdot \mathbf{B}}{2\pi} \]

Real: \[ \frac{dR}{dt} = \frac{dR}{dt} = \frac{dR}{dt} = -\frac{R \cdot dB}{2\pi} \Rightarrow 2\frac{dR}{dt} = \frac{dB}{B} \]

v. Thus \( R^2B = \text{const.} \)

vi. The Magnetic Flux through drift orbit is

\[ \Phi_B = \pi R^2B = \text{const.} \]

(Assuming \( B \) is relatively constant near axis of symmetry.)

vii. Thus, the 3rd Adiabatic Invariant means the flux enclosed by drift orbit remains constant.

Particle remains on the surface of a flux tube.

E. Example: Magnetosphere

\( v_e \)\n
![Larmor Motion]

![Nested drift motion (Ring Current)]

1. Second Adiabatic Invariant applies even without axisymmetry.
2. a. For a constant Magnetic field (\( \frac{dB}{dt} = 0 \)), energy is conserved.
   \[ \mathcal{E} = \frac{1}{2}mv_{\perp}^2 + \mu B(\mathbf{s}) \]
   b. Since \( \mathcal{E} = \text{const.} \), \( B_\perp = \text{constant} \).
   c. Freeing \( v_{\perp} \) at \( B_\perp \), \( \mathcal{I} = \int_{B_\perp} B_\perp \mathbf{s} \mathbf{d}s = \text{const.} \)

3. Higher order multiple manifolds:
   a. Conserve \( B_\perp \) & \( \mathcal{I} \) mean that drifting particles remain on a surface (L-shell).