Lecture #9: Particle Motion in Slowly Varying $\underset{\sim}{E}$-fields Howes (1)
Polarization Drift

I. Polarization Drift

A. Consider an Electric field varying slowly in time $\underset{\sim}{E}(\tau)$ with a constant Magnetic field $\underset{\sim}{B} = B_0 \hat{\underset{\sim}{z}}$.

1. NOTE: $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \epsilon_0 \mu_0 \frac{\partial \underset{\sim}{E}}{\partial t}$, we assume $|\underset{\sim}{j}| \gg |\epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}|$, so $\frac{d\underset{\sim}{B}}{dt} \simeq 0$.

B. Multiple Time Scale Analysis

1. $\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

2. a. Take $\underset{\sim}{E}(\tau)$ varies only on slow timescale $\tau = \epsilon t$
   b. $\underset{\sim}{v} = \underset{\sim}{v}_1(t) + \epsilon \underset{\sim}{v}_2(\tau) + \epsilon^2 \underset{\sim}{v}_3(\tau) + \dots$
   c. Also assume $\underset{\sim}{E}(\tau) \cdot \underset{\sim}{B} = 0$

3. As before, $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$

4. We'll take a small electric field such that the $E \times B$ drift velocity $v_E \ll v$, where v is Larmor orbit velocity.
   Thus $\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\epsilon \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

5. Substitute expanded solution:

$$\frac{\partial}{\partial t}\left(\underset{\sim}{v}_1 + \cancelto{0}{\epsilon \underset{\sim}{v}_2} + \cancelto{0}{\epsilon^2 \underset{\sim}{v}_3}\right) + \epsilon \frac{\partial}{\partial \tau}\left(\cancelto{0}{\underset{\sim}{v}_1} + \epsilon \underset{\sim}{v}_2 + \epsilon^2 \underset{\sim}{v}_3\right) = \epsilon \frac{q}{m}\underset{\sim}{E}(\tau)$$
$$+ \frac{q}{m}\left(\underset{\sim}{v}_1 + \epsilon \underset{\sim}{v}_2 + \epsilon^2 \underset{\sim}{v}_3\right) \times \underset{\sim}{B}$$

a. Taking $\underset{\sim}{B} = B_0 \hat{\underset{\sim}{b}}$,

$$\frac{\partial \underset{\sim}{v}_1}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{v}_2}{\partial \tau} + \epsilon^3 \frac{\partial \underset{\sim}{v}_3}{\partial \tau} = \epsilon \frac{q \underset{\sim}{E}(\tau)}{m} + \omega_c \underset{\sim}{v}_1 \times \hat{\underset{\sim}{b}} + \epsilon \omega_c \underset{\sim}{v}_2 \times \hat{\underset{\sim}{b}} + \epsilon^2 \omega_c \underset{\sim}{v}_3 \times \hat{\underset{\sim}{b}}$$

I.B. (Continued)

6. $\mathcal{O}(1)$: $\frac{\partial \mathbf{v}_1}{\partial t} = \omega_c \mathbf{v}_1 \times \hat{\mathbf{b}}$

a. This is just the usual, fast timescale Larmor gyration about the magnetic field.

b. The general solution for this motion can be written

$$\boxed{\mathbf{v}_1 = v_\perp \cos(\omega_c t + \phi)\hat{\mathbf{e}}_1 - v_\perp \sin(\omega_c t + \phi)\hat{\mathbf{e}}_2 + v_{\parallel}\hat{\mathbf{b}}}$$

for a right-handed coordinate system s.t. $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$

7. $\mathcal{O}(\epsilon)$: $0 = \cancel{\epsilon}\frac{q}{m}\mathbf{E}(\tau) + \cancel{\epsilon}\frac{qB_0}{m}\mathbf{v}_2 \times \hat{\mathbf{b}}$

a. This is just the slow timescale $\mathbf{E}\times\mathbf{B}$ drift.

b. Operating $\hat{\mathbf{b}}\times$ on equation gives:

$$\frac{q}{\cancel{m}}\hat{\mathbf{b}}\times\mathbf{E}(\tau) = \frac{qB_0}{\cancel{m}}\hat{\mathbf{b}}\times(\mathbf{v}_2\times\hat{\mathbf{b}}) = B_0\left(\mathbf{v}_2(\hat{\mathbf{b}}\cdot\hat{\mathbf{b}}) - v_{2\parallel}\hat{\mathbf{b}}\right)$$

or

$$\boxed{\mathbf{v}_2 = v_{2\parallel}\hat{\mathbf{b}} + \frac{\mathbf{E}(\tau)\times\mathbf{B}}{B_0^2}}$$

8. $\mathcal{O}(\epsilon^2)$: $\cancel{\epsilon^2}\frac{\partial \mathbf{v}_2}{\partial \tau} = \cancel{\epsilon^2}\omega_c \mathbf{v}_3 \times \hat{\mathbf{b}}$

a. At this order, the solution $\mathbf{v}_2$ is considered to be known.

Thus, $\frac{\partial \mathbf{v}_2}{\partial \tau} = \cancel{\frac{\partial v_{2\parallel}}{\partial \tau}}^{\,0}\hat{\mathbf{b}} + \frac{1}{B_0^2}\frac{\partial \mathbf{E}}{\partial \tau}\times\mathbf{B} = \frac{\frac{\partial \mathbf{E}}{\partial \tau}\times\hat{\mathbf{b}}}{B_0}$

b. $\frac{1}{B_0}\frac{\partial \mathbf{E}}{\partial \tau}\times\hat{\mathbf{b}} = \omega_c \mathbf{v}_3 \times \hat{\mathbf{b}}$

c. Take $\hat{\mathbf{b}}\times$ this equation

$$\frac{1}{B_0}\hat{\mathbf{b}}\times\left(\frac{\partial \mathbf{E}}{\partial \tau}\times\hat{\mathbf{b}}\right) = \frac{1}{B_0}\left[\frac{\partial \mathbf{E}}{\partial \tau}(\hat{\mathbf{b}}\cdot\hat{\mathbf{b}}) - \cancel{\hat{\mathbf{b}}\left(\hat{\mathbf{b}}\cdot\frac{\partial \mathbf{E}}{\partial \tau}\right)}^{\,0}\right] = \frac{1}{B_0}\frac{\partial \mathbf{E}}{\partial \tau}$$

$$\omega_c \hat{\mathbf{b}}\times(\mathbf{v}_3\times\hat{\mathbf{b}}) = \omega_c\left[\mathbf{v}_3(\hat{\mathbf{b}}\cdot\hat{\mathbf{b}}) - \hat{\mathbf{b}}(\mathbf{v}_3\cdot\hat{\mathbf{b}})\right] = \omega_c\left(\mathbf{v}_3 - v_{3\parallel}\hat{\mathbf{b}}\right)$$

d. Thus

$$\boxed{\mathbf{v}_3 = v_{3\parallel}\hat{\mathbf{b}} + \frac{1}{\omega_c B_0}\frac{\partial \mathbf{E}(\tau)}{\partial \tau}}$$

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I. B. (Continued)

9. Putting the full solution together: (Taking $v_{2\parallel} = v_{3\parallel} = 0$)

$$\boxed{\underset{\sim}{v} = \underbrace{v_\perp\left(\cos(\omega_c t+\phi)\hat{\underset{\sim}{e}}_1 - \sin(\omega_c t+\phi)\hat{\underset{\sim}{e}}_2\right)}_{\text{Zeroth order Larmor Motion}} + \underbrace{\frac{\underset{\sim}{E}(t)\times\underset{\sim}{B}}{B^2}}_{\text{First-order } \underset{\sim}{E}\times\underset{\sim}{B} \text{ drift}} + \underbrace{\frac{1}{\omega_c B_0}\frac{d\underset{\sim}{E}}{dt}}_{\text{Second-order Polarization Drift}}}$$

C. Polarization Drift:

1. For slowly varying electric field $\underset{\sim}{E}(t)$ (slow with respect to the Larmor motion), ~~then~~ we define the

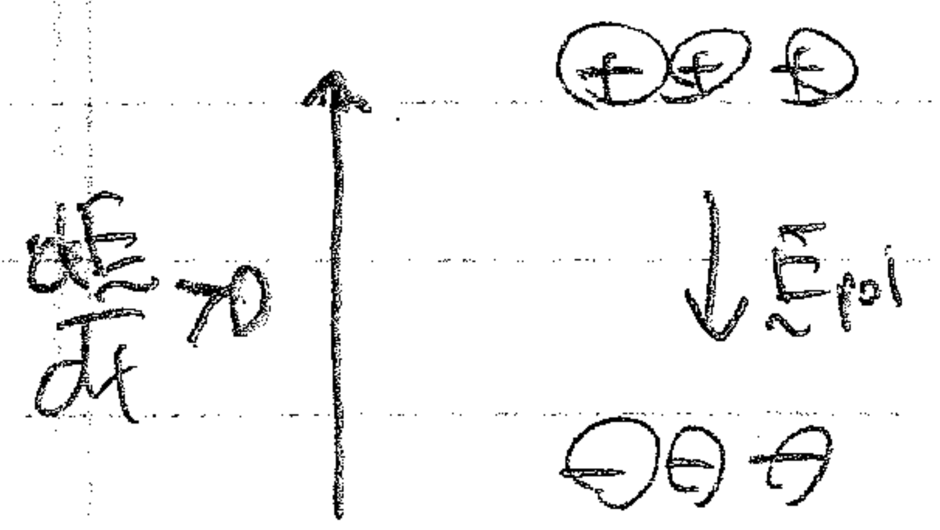
Polarization Drift $\boxed{\underset{\sim}{v}_p \equiv \frac{1}{\omega_c B}\frac{d\underset{\sim}{E}}{dt}}$

2. Using $\omega_c = \frac{qB}{m}$, we have

$$\underset{\sim}{v}_p = \frac{m}{qB^2}\frac{d\underset{\sim}{E}}{dt}$$

a. Polarization drift is charge dependent

$\Rightarrow$ ions and electrons drift in opposite directions



b. Resulting polarization of plasma opposes increasing applied Electric Field.

c. Because $m_i \gg m_e$, ions dominate the polarization drift.

3. Polarization Current: $\underset{\sim}{j}_p = \sum_s q_s n_s \underset{\sim}{v}_{ps} = \sum_s \frac{q_s n_s m_s}{q_s B^2}\frac{d\underset{\sim}{E}}{dt}$

a. $\underset{\sim}{j}_p = \sum_s \frac{n_s m_s}{B^2}\frac{d\underset{\sim}{E}}{dt}$

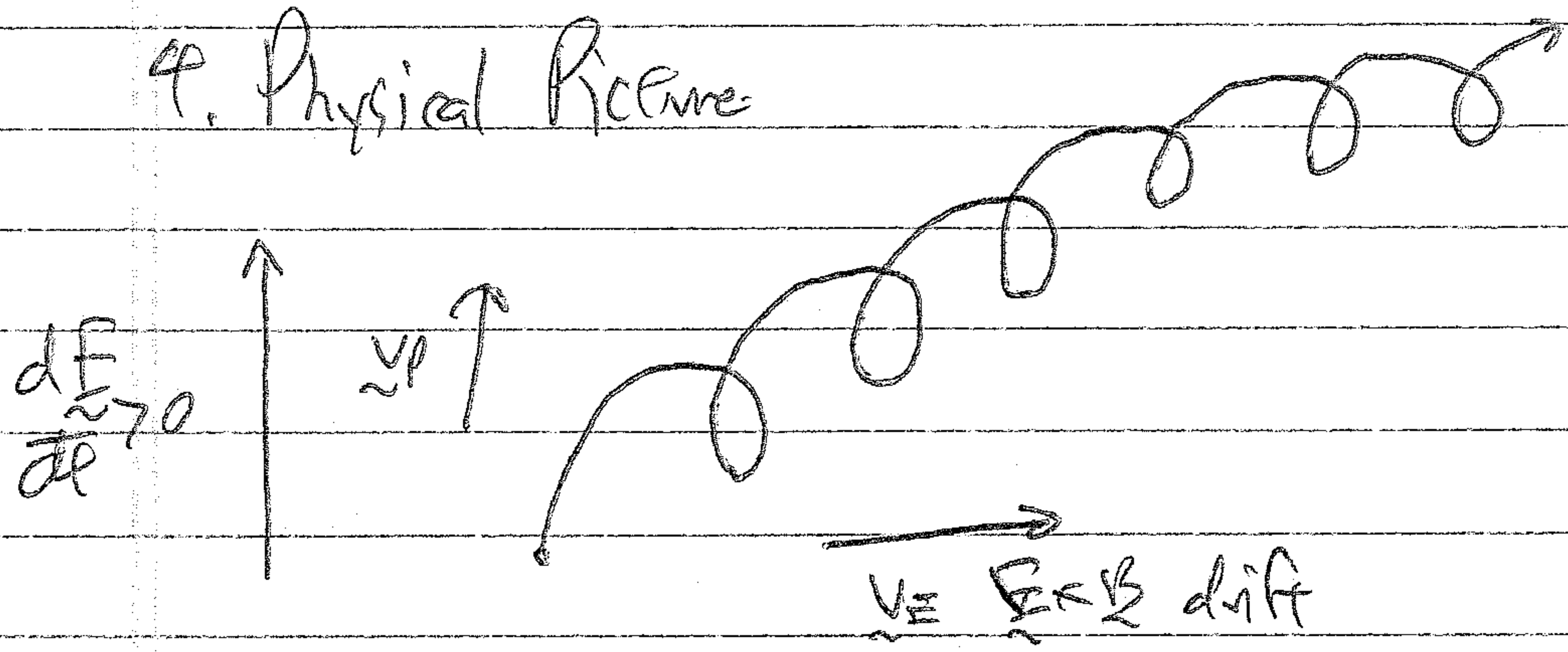
b. Mass dependence means ion contributes most to polarization current.

I.C.3. (Continued)

b. NOTE: $\mathbf{E}\times\mathbf{B}$ velocity is the same for both species, so it cancels, producing no net current.

$$\mathbf{j}_E = \sum_s q_s n_s \mathbf{v}_E = \sum_s q_s n_s \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}\underbrace{\cancel{\sum_s q_s n_s}}_{0 \text{ by quasineutrality}} = 0$$

4. Physical Picture



5. The Polarization Drift can lead to an increase in energy.

a. $$\frac{d\varepsilon}{dt} = \mathbf{v}\cdot\mathbf{F} = \mathbf{v}\cdot q(\mathbf{E} + \cancel{\mathbf{v}\times\mathbf{B}}) = q\,\mathbf{v}\cdot\mathbf{E}$$

b. $$= q\Big[\underbrace{\cancel{v_\perp \cos(\omega_c t+\phi)\,\mathbf{E}\cdot\hat{\mathbf{e}}_1 + v_\perp \sin(\omega_c t+\phi)\,\mathbf{E}\cdot\hat{\mathbf{e}}_2}}_{\text{Average over Larmor orbit} \Rightarrow 0} + \underbrace{\cancel{\frac{\mathbf{E}(t)\times\mathbf{B}}{B^2}\cdot\mathbf{E}}}_{0} + \frac{1}{\omega_c B}\frac{d\mathbf{E}}{dt}\cdot\mathbf{E}\Big]$$

c. Thus $$\frac{d\varepsilon}{dt} = \frac{q}{\omega_c B}\frac{d}{dt}\left(\frac{E^2}{2}\right) = \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{E^2}{B^2}\right)\right]$$

d. NOTE: $|\mathbf{v}_E|^2 = \frac{E^2}{B^2}$, so this can be written $\boxed{\frac{d}{dt}\left(\frac{1}{2}m v_E^2\right) = \frac{d\varepsilon}{dt}}$

e. $\boxed{\text{The Polarization Drift leads to the acceleration of particles to achieve the } \mathbf{E}\times\mathbf{B} \text{ drift velocity.}}$