

29:195 Homework #4

Reading: Required: Read GB Chapter 8, Sections 8.1–8.3 (p.281–311)
Optional: Read BS Chapter 7, Sections 7.1–7.3 (p.252–268)

Due at the beginning of class, Thursday, March 12, 2009.

1. A plasma has a “spherical shell” distribution function given by

$$f_0(\mathbf{v}) = \frac{n_0}{4\pi C^2} \delta(|\mathbf{v}| - C)$$

where C is a constant.

- (a) Using the Fourier analysis approach, show that the dispersion relation for electrostatic waves in this plasma is $\omega^2 = \omega_p^2 + k^2 C^2$.
- (b) What is the region of validity of this dispersion relation?
2. Show that the Laplace transform of $f(t) = \cosh(at)$ is given by

$$\tilde{f}(p) = \frac{p}{p^2 - a^2}.$$

3. Use the Residue Theorem to evaluate the inverse Laplace transform of

$$\tilde{f}(p) = \frac{1}{p^2 - a^2}.$$

4. Solution of Navier-Stokes Equations:

The Navier-Stokes Equations for the viscous evolution of a hydrodynamic fluid are given by:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho &= -\rho \nabla \cdot \mathbf{U} \\ \rho \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) &= -\nabla p + \nu \rho \nabla^2 \mathbf{U} \\ \frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p &= -\gamma p \nabla \cdot \mathbf{U} \end{aligned}$$

where ν is the coefficient of kinematic viscosity. Assume a wave vector of the form $\mathbf{k} = k\hat{\mathbf{z}}$. The initial conditions for this system at $t = 0$ are $\mathbf{U}(\mathbf{x}, 0) = U_0 \cos(kz)\hat{\mathbf{z}}$ and $\mathbf{U}'(\mathbf{x}, 0) = -U_0\omega_0 \sin(kz)\hat{\mathbf{z}}$.

- (a) Using the Laplace-Fourier transform method (Fourier transform in space, Laplace transform in time) to solve for the velocity $\mathbf{U}(\mathbf{x}, t)$ in complex terms. Be sure to use the definition $c_s^2 = \gamma p_0 / \rho_0$ to simplify your calculations.
- (b) Determine the evolution of the magnitude of the velocity $|\mathbf{U}(\mathbf{x}, t)|$.
- (c) In the weak damping limit $\nu^2 k^2 \ll 4c_s^2$, what are the effective real frequency of oscillation (include the small, first order correction) and damping rate?
- (d) Qualitatively sketch the solution $\mathbf{U}(z = 0, t)$ in the case that $\nu^2 k^2 < 4c_s^2$.
- (e) Qualitatively sketch $\mathbf{U}(z = 0, t)$ for the cases $\nu^2 k^2 = 4c_s^2$ and $\nu^2 k^2 > 4c_s^2$ on the same plot (but a different plot from part d).