

## 29:195 Homework #5

Reading: Required: Read GB Chapter 8, Sections 8.4–8.5 (p.311–334)  
 Optional: Read BS Chapter 7, Sections 7.4–7.5 (p.268–277)

Due at the beginning of class, Thursday, March 26, 2009.

1. We have shown in class that the Cauchy distribution

$$F_{0C}(v_z) = \frac{C}{\pi} \frac{1}{C^2 + v_z^2}$$

yields a dispersion relation

$$D(\mathbf{k}, p) = 1 + \frac{\omega_p^2}{(p + |k|C)^2}$$

which has a solution  $\omega = \pm\omega_p$  and  $\gamma = -|k|C$ . Show that in the high phase velocity limit ( $|k| \rightarrow 0$ ), the weak growth rate approximation gives the same result.

2. Large Argument Expansion of the Plasma Dispersion Function

- (a) If the pole at  $\xi = x + iy$  is very close to the  $z$  axis ( $|y| \ll |x|$ ), show that the plasma dispersion function is given by

$$Z(\xi) = i \frac{k}{|k|} \sqrt{\pi} e^{-\xi^2} + \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} \frac{e^{-z^2}}{z - \xi} dz.$$

- (b) By writing

$$\frac{1}{z - \xi} = \frac{-1}{\xi(1 - z/\xi)} = -\frac{1}{\xi} \left[ 1 + \left(\frac{z}{\xi}\right) + \left(\frac{z}{\xi}\right)^2 + \dots \right]$$

and integrating term by term, show that in the limit of large  $\xi$  the plasma dispersion function is given by the following power series

$$Z(\xi) = i \frac{k}{|k|} \sqrt{\pi} e^{-\xi^2} - \left[ \frac{1}{\xi} + \frac{1}{2\xi^3} + \frac{3}{4\xi^5} + \dots \right].$$

3. Using the Error Function representation of the plasma dispersion function

$$Z(\xi) = i\sqrt{\pi} e^{-\xi^2} [1 + \operatorname{erf}(i\xi)]$$

where

$$\operatorname{erf}(i\xi) = \frac{2}{\sqrt{\pi}} \int_0^{i\xi} e^{-z^2} dz,$$

show that for small  $\xi$ ,

$$Z(\xi) = i\sqrt{\pi} e^{-\xi^2} - 2\xi + \frac{4}{3}\xi^3 - \frac{8}{15}\xi^5 + \dots$$

Hint: Use a Taylor Series expansion for  $\exp(-\xi^2)$  and  $\exp(-z^2)$ , integrate term by term, and then collect like powers of  $\xi$ .

4. For a plasma consisting of protons and electrons, both with Maxwellian velocity distributions, the dispersion relation can be written

$$D(\mathbf{k}, p) = 1 - \frac{1}{k^2 \lambda_{De}^2} \frac{1}{2} \left[ Z'(\xi_e) + \frac{T_e}{T_i} Z'(\xi_i) \right] = 0,$$

where  $T_e$  is the electron temperature,  $T_i$  is the ion temperature, and the derivative of the Plasma Dispersion Function is denoted by  $Z'(\xi) = \partial Z(\xi)/\partial \xi$ .

- (a) Use the large-argument expansion of the plasma dispersion function for the ions and the small argument expansion for the electrons to simplify the dispersion relation and obtain the analytical solutions

$$\frac{\omega}{k} = \pm \sqrt{\frac{T_e}{m_i}} \frac{1}{(1 + k^2 \lambda_{De}^2)^{1/2}},$$

and

$$\gamma/\omega = -\sqrt{\frac{\pi}{8}} \left[ \sqrt{\frac{m_e}{m_i}} + \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e}{2T_i} \frac{1}{(1 + k^2 \lambda_{De}^2)}\right) \right] \frac{1}{(1 + k^2 \lambda_{De}^2)^{3/2}}.$$

- (b) In what limit of the real frequency  $\omega$  is this solution valid?

5. (a) To model a hot beam, one can use a shifted Cauchy distribution of the form

$$F_0(v_z) = \frac{C}{\pi} \frac{1}{C^2 + (v_z - U)^2}$$

where  $U$  is the beam velocity. Show that the dispersion relation for this plasma is

$$D(\mathbf{k}, p) = 1 + \frac{\omega_p^2}{(p + |k|C + ikU)^2} = 0$$

- (b) Solve for the real frequency and damping rate of such a plasma.