

29:195 Homework #7

Due at the beginning of class, Thursday, May 7, 2009.

1. Hydrodynamic Keplerian Accretion Disk

Calculate the dispersion relation for a hydrodynamic disk in Keplerian rotation about a central body of mass M . Assume incompressible motion $\nabla \cdot \mathbf{U} = 0$ and a wave vector $\mathbf{k} = k\hat{\mathbf{z}}$ that varies only in the z direction (aligned with the axis of the Keplerian rotation). Take the accretion disk to be an isothermal, thin disk.

- Write down the relevant first-order hydrodynamic equations (having removed the equilibrium) based on the assumptions above.
- Why do the pressure gradient and gravitational force terms in the momentum equation not contribute to the first order equations?
- Show that the dispersion relation for this system is

$$\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d \ln R}.$$

- Use the definition of the epicyclic frequency κ to show that this dispersion relation may be alternatively written as $\omega^2 = \kappa^2$.
- Show that this implies a stability criterion $dL/dR > 0$ for stability and that the Keplerian disk is stable. Here $L = R^2\Omega$ is the specific angular momentum.

2. Growth Rates of the Magnetorotational Instability

In a magnetized Keplerian accretion disk, the dispersion relation for fluctuations with $\mathbf{k} = k\hat{\mathbf{z}}$ in the incompressible limit is

$$\omega^4 - \omega^2(\kappa^2 + 2k^2v_A^2) + k^2v_A^2 \left(k^2v_A^2 + \frac{d\Omega^2}{d \ln R} \right) = 0.$$

- Determine the maximum unstable growth rate $\gamma_{max} = \text{Im}(\omega)$ for an arbitrary unstable rotation profile $\Omega(R)$ with $d\Omega/dR < 0$.
- Calculate the wavenumber $(kv_A)_{max}^2$ at which this maximum growth rate occurs.
- For a Keplerian rotation profile $\Omega^2 = GM/R^3$, calculate the values of γ_{max} and $(kv_A)_{max}$ in terms of the angular rotation frequency Ω .

3. Hydrodynamic Turbulence

If a cup of coffee has a radius of about 5 cm, and it takes 2 seconds to stir around the cup once, estimate the time it takes for the turbulence to reach the viscous scale of approximately 10^{-2} cm.

4. MHD Turbulence in the Solar Wind

Although the solar wind is actually a collisionless plasma, it has been rigorously shown that the dynamics of Alfvén wave turbulence is well described by MHD at scales larger than the ion Larmor radius r_{Li} . If the turbulence in the solar wind is driven isotropically ($L = L_{\perp} = L_{\parallel}$) at a scale of 10^{12} cm with a velocity equal to the local Alfvén velocity, estimate the anisotropy (k_{\parallel}/k_{\perp}) of critically balanced fluctuations when the perpendicular scale of the ion Larmor radius is reached, $k_{\perp}r_{Li} = 1$. The parameters of the solar wind plasma are $B_0 \sim 10^{-4}$ G, $T_i = T_e \sim 5 \times 10^4$ K, and $n_i = n_e \sim 20 \text{ cm}^{-3}$. (In this calculation, do keep factors of 2π).