Lecture #10: Drift Waves

I. Drift Waves

A. General Comments

1. Our investigation of plasma waves thus far has focused on infinite, uniform plasmas with a straight magnetic field. However, most plasmas about which we care are confined, and therefore have density gradients.

2. An important class of waves that exist only in plasmas with a density or temperature gradient are **Drift Waves**.

B. Drift Waves in a Plasma with a Density Gradient

1. **Low Beta Plasma:** \( \frac{m_e}{m_i} \ll \beta_e \ll 1 \) where \( \beta_e = \frac{2 m_e n_e T_e}{B_0^2} \)

   a. Here magnetic pressure dominates over thermal pressure.

2. a. \( B_0 = B_0 \hat{z} \) \( E_0 = 0 \) *straight, uniform \( B_0 \) density gradient

   b. \( n_0 \equiv N_0 = N_0(x) \)

   c. \( T_e = T_0 = constant \) *isothermal electrons*

   d. \( T_i = 0 \) *cold ions*

   Thus, the electron eq. of state is \( \boxed{P_e = n_e T_e} \) \( \gamma_e = 1 \)

   

In these limits, we will solve for **Electron Drift Waves** & into \( T_e \)

3. Geometry:

\[
\nabla n_0
\]

\[
B_0 = B_0 \hat{z}
\]
Lecture #10 (Continued)
2. B. (Continued)

4. Two-Fluid Treatment: (See Lecture #14, III)
   a. In this limit, the two fluid system is

   Continuity: \( \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{U}_i) = 0 \)

   \[ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{U}_e) = 0 \]

   Momentum: \( n_i \mathbf{U}_i = \rho_i (\mathbf{E} + \mathbf{U}_i \times \mathbf{B}) \)

   \[ n_e \mathbf{U}_e = \rho_e (\mathbf{E} + \mathbf{U}_e \times \mathbf{B}) \]

   Eq. of State:
   \( p_i = T_i = 0 \)

   \( p_e = N_e T_e \)

   Poisson's Eq:
   \( \nabla \cdot \mathbf{E} = \frac{\rho_i}{\varepsilon_0} \)

   \( \rho_i = n_i \varepsilon_0 \frac{q_i}{m_i} \)

   Faraday's law:
   \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)

   \( j = \varepsilon_0 n_i \frac{q_i}{m_i} \mathbf{E} \)

   Ampere/Maxwell law:
   \( \nabla \times \mathbf{E} = \mu_0 j + \frac{\mu_0}{c} \frac{\partial \mathbf{B}}{\partial t} \)

   \( \nabla \cdot \mathbf{B} = 0 \)

C. Equilibrium:

1. Ordeding:
   \( n_i = n_i^{(0)} + c n_i \)

   \( n_e = n_e^{(0)} + c n_e \)

   \( \mathbf{U}_i = \mathbf{U}_i^{(0)} + \epsilon \mathbf{U}_i \)

   \( \mathbf{U}_e = \mathbf{U}_e^{(0)} + \epsilon \mathbf{U}_e \)

   \( \mathbf{E} = \mathbf{E}_0^{(0)} + \epsilon \mathbf{E} \)

   \( \mathbf{B} = \mathbf{B}_0^{(0)} + \epsilon \mathbf{B} \)

   \( \mathbf{E}_0 = 0 \) for equilibrium "0" quantities.

2. Electron Momentum Eqn:

   a. \( 0 = \varepsilon_0 n_e \mathbf{E}_{e_0} \cdot \nabla \mathbf{E}_{e_0} = -\nabla \cdot \mathbf{P}_{e_0} + q_e n_e (\mathbf{E}_{e_0} \times \mathbf{B}_0) \)

   \( \nabla \cdot \mathbf{P}_{e_0} = -q_e n_e (\mathbf{E}_{e_0} \times \mathbf{B}_0) \)

   \( n_{e_0} = \frac{\partial n_{e_0}}{\partial x} \)

   b. \( \nabla \cdot \mathbf{P}_{e_0} = -Te V_{e_0} \mathbf{V}_{e_0} = -Te \frac{\partial n_{e_0}}{\partial x} \mathbf{V}_{e_0} \)

   c. For small electron mass, we can neglect LHS, leaving

   \( 0 = -Te n_{e_0} \mathbf{V}_{e_0} + q_e n_{e_0} B_0 \mathbf{U}_e \times \mathbf{V}_{e_0} \)

   d. By taking \( \mathbf{V}_{e_0} \), we solve for \( \mathbf{U}_e \):

   \( \mathbf{U}_e = \frac{\mathbf{V}_{e_0}}{q_e B_0} \left( \frac{\partial n_{e_0}}{\partial x} \right) \mathbf{v} + U_{e_0} \mathbf{v} \)

   d. Taking \( U_{e_0} = 0 \), the drift velocity is

   Equilibrium Drift Velocity: \( \mathbf{U}_e = \frac{\mathbf{V}_{e_0}}{q_e B_0} \left( \frac{\partial n_{e_0}}{\partial x} \right) \mathbf{v} \)
3. **Define: Drift Velocity**

\[ U_d = \frac{T_e}{q_e B_0} \left( \frac{\partial n_o}{\partial z} \right) \]

4. **Note:** This is just the usual drift due to a general force \( \mathbf{F} \) density \( \mathbf{F} = \frac{\mathbf{E} \times \mathbf{B_0}}{q_n B_0^2} \) where the Lorentz electron pressure gradient \( \mathbf{F} = -\nabla p_e \).

5. **Equilibrium Picture:**

   This is a perfectly good equilibrium that maintains a steady-state drift in the \( \hat{x} \)-direction, stable with gravity when \( \nabla n_o < 0 \) or, for any \( \nabla n \) when some density \( \mathbf{F} = q_e n_o \mathbf{g} \approx -\nabla p_e \)

6. **Note:** Since \( T_i = 0 \), \( U_{i0} = 0 \), Ions do not drift (no pressure force).

7. **Low Frequency Wave Solutions**

   a. We know for Alfvén Waves in Uniform Plasma, \( c_A = \pm K_{ii} V_A \).

   b. In this limit, the magnetic field is not perturbed, \( B_i = 0 \).

   Faraday's Law: \( \frac{\partial \mathbf{E}}{\partial t} = -\nabla \mathbf{\phi} \Rightarrow \mathbf{c_A} \mathbf{B_i} = \mathbf{K_{ii}} \mathbf{E_i} \Rightarrow \mathbf{K_{ii}} \mathbf{E_i} = 0 \Rightarrow \text{Electrostatic} \)

   c. For Electrostatic Perturbations, we may take \( \mathbf{E} = -\nabla \phi \)

2. **Note:** We'll also assume \( c_A \ll c_{ci} \) Low Frequency compared to ion cyclotron freq.
3. Boltzmann Distribution for Electrons
   a. Electron Momentum:

   \[ O(c): \rho e n_0 [U_{e0} \nabla U_{e1} + U_{e1} \nabla U_{e0}] = -\nabla p_{e1} - \rho e n_0 \nabla \Phi_i + \rho n e \left( U_{e0} \nabla i + U_{e1} \nabla j + U_{e2} \nabla k \right) \]

   b. For electrostatic perturbations, \( B = 0 \).

   c. Again, we treat the electron mass as very small \( \Rightarrow \) LHS = 0

   d. Thus, we find

   \[ 0 = -\nabla n_{e1} + \rho n e \nabla \left( U_{e1} + k_B T \right) - \rho e n_0 \nabla \phi_i \]

   e. Taking the product with \( \frac{\hat{z}}{z} \):

   \[ \hat{z} \cdot \left( U_{e1} \frac{\hat{z}}{z} \right) = 0, \quad \text{so} \]

   \[ T \frac{\partial n_{e1}}{\partial z} = -\rho e n_0 \frac{\partial \phi_i}{\partial z} \]

   f. Integrating over \( z \):

   \[ \int \frac{\partial n_{e1}}{\partial z} dz = \int \frac{-\rho e n_0}{T} \frac{\partial \phi_i}{\partial z} dz \Rightarrow \ln n_{e1} = \frac{-\rho e \phi_i}{T} + \text{const}. \]

   \[ \Rightarrow \quad n_{e1} = n_0 e^{-\frac{\rho e \phi_i}{T}} \quad \text{Boltzmann Distribution.} \]

   Linearized:

   \[ e^{-\frac{\rho e \phi_i}{T}} \approx 1 - \frac{\rho e \phi_i}{T} \Rightarrow \quad n_{e1} \approx n_0 \left( 1 - \frac{\rho e \phi_i}{T} \right) \]

   \[ \Rightarrow \quad n_{e1} = -n_0 \frac{\rho e \phi_i}{T} \]

9. Physically, the very low mass electrons move along field lines much more rapidly than the wave, thermalizing and giving a Boltzmann distribution. Thus, isothermal approximation \( T = \text{const} \) is consistent.

4. Simplification: Take \( i \sim e^{i(k_y y + k_z z - \omega t)} \) \( \Rightarrow k_y^2 \approx 0 \) (in \( y-z \) plane)

5. Ion Momentum Equation (Remember \( U_{e0} = 0 \))

a. \( O(c): \rho i_{e0} \frac{\partial U_{i1}}{\partial x} = -q_i \rho \nabla \phi_i + q_i \rho n_0 \nabla \cdot (U_{i1} \nabla) \)

b. \[ -i \omega U_{i1} = -\frac{q_i}{m_i} k \phi_i + \frac{q_i B_0}{m_i} U_{i1}^{\cdot \cdot \cdot \cdot} \]
\[ c. \quad \omega U_{il} = \pm \frac{q_i}{m_i} k_l \phi_l + i \alpha_k \cdot \mathbf{U}_{il} \times \mathbf{k} \]

1. Solving for \( U_{il} \) in terms of \( \phi_l \):
\[
\begin{align*}
\omega U_{lx} &= i \alpha_c U_{ly} \\
\omega U_{ly} &= \frac{q_i}{m_i} k_y \phi_l - i \alpha_k U_{lx} \\
\omega U_{lz} &= \frac{q_i}{m_i} k_z \phi_l
\end{align*}
\]

\[ U_{lx} = \frac{i \alpha_c q_i}{m_i} k_y \phi_l \\
U_{ly} = \frac{\omega - \alpha_c^2}{\omega - \alpha_k^2} \frac{q_i}{m_i} k_y \phi_l \\
U_{lz} = \frac{q_i}{m_i} k_z \phi_l \]

6. **Ion Continuity**:

a. \( \nabla \cdot \mathbf{u}_{ii} + U_{ii} \cdot \nabla n_0 + n_0 \nabla \cdot U_{ii} = 0 \)

b. \( \frac{\partial n_{ii}}{\partial t} + \frac{\nabla}{\nabla_i} \left( \frac{q_i}{m_i} U_{ix} + \frac{q_i}{m_i} \frac{k_y U_{iy}}{\omega} + \frac{q_i}{m_i} \frac{k_z U_{iz}}{\omega} \right) = 0 \)

c. Substituting in for \( U_{ii} \) to yield \( n_{ii} \) in terms of \( \phi_l \):
\[
\frac{n_{ii}}{n_0} = \frac{\omega}{\omega^2 - \alpha_k^2} \frac{\omega}{\alpha_k^2} \phi_l + \frac{q_i}{m_i} \frac{\alpha_k^2}{\omega^2 - \alpha_k^2} \phi_l + \frac{q_i}{m_i} \frac{k_z^2}{\omega} \phi_l
\]

\[ \frac{n_{ii}}{n_0} = -\frac{k_y}{\omega} \left( \frac{\omega}{\alpha_k^2} \right) \phi_l + \frac{q_i}{m_i} \frac{k_z^2}{\omega} \phi_l = \left( \frac{k_y}{\omega} \frac{\omega}{\alpha_k^2} \left( \frac{\omega}{\alpha_k^2} \right) + \frac{q_i}{m_i} \frac{k_z^2}{\omega} \right) \left( \phi_l \right) \]

\[ U_{ld} \]

e. Thus,
\[
\frac{n_{ii}}{n_0} = \left( \frac{k_y U_d + k_z^2 \ell^2}{\omega + \frac{k_z^2}{\omega}} \right) \left( \frac{\omega}{\alpha_k^2} \phi_l \right)
\]

where we recall the **Ion Acoustic Speed** \( C_i^2 = \frac{T_e}{m_i} \)

From 29194 Lea\#24 (II, Fig.), Again, we absorb Boltzmann's constant \( k \) into temperature \( T_e \) to give temperature in energy units.

(to avoid confusion with the wavenumber \( k \) )
Lecture 10 (Continued)
E. D. Continued

7. Now we have \( n_1 = f(\phi_1) \), \( n_1 = f(\phi_2) \), so we can use Poisson's Equation to solve for linear dispersion relation.

\[ \nabla \cdot E = \frac{P_0}{\varepsilon_0} = \frac{n_i q_i + n_e q_e}{\varepsilon_0} \]

(Note: \( n_i q_i + n_e q_e = 0 \)
(Neutral Equilibrium)

b. Using \( E = -\nabla \phi \) and linearizing:

\[ \mathcal{E}(\varepsilon) \setminus -\nabla^2 \phi \approx \frac{n_i q_i + n_e q_e}{\varepsilon_0} \Rightarrow \kappa^2 \phi_1 = \frac{n_i q_i + n_e q_e}{\varepsilon_0} \]

\[ \kappa^2 \phi_1 = \frac{n_0 q_i^2}{\varepsilon_0 e} \left( \frac{k_i U_d + k_i^2 c_i^2}{\omega^2} \right) \phi_1 + \frac{n_0 q_e^2}{\varepsilon_0 e} \phi_1 \]

d. Multiplying by \( \frac{T_i}{m_i} \) yields:

\[ \kappa^2 c_i^2 = \omega p_i^2 \left( \frac{k_i U_d + k_i^2 c_i^2}{\omega^2} \right) - \omega p_i^2 \]

Eventually, we obtain:

\[ 1 + \frac{\omega p_i^2}{\kappa^2 c_i^2} \left( \frac{\omega^2 - \omega k_i U_d + k_i^2 c_i^2}{\omega^2} \right) = 0 \]

Electron Drift Wave Dispersion Relation
(Low Frequency Limit)

E. Long Wavelength Drift Waves

1. For long wavelengths \( \kappa^2 c_i^2 \ll \omega p_i^2 \), the dispersion relation simplifies:

\[ \omega^2 - \omega k_i U_d - k_i^2 c_i^2 = 0 \]

2. Solution:

\[ \omega = k_i U_d \left[ \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{4k_i^2 c_i^2}{k_i U_d^2}} \right] \]

3. Limits:

a. \( U_d \rightarrow 0 \) (\( \nu_0 = 0 \)) \( \omega^2 = k_i^2 c_i^2 \) Ion Acoustic Waves (except 24)

b. \( k_i \rightarrow 0 \)

i. \( \omega = k_i U_d \) Drifting Plasma Oscillations

ii. \( \omega = -\frac{k_i^2 c_i^2}{k_i U_d} \rightarrow 0 \)
4. Dispersion Relation: Fixed $k_1$, $\omega$ vs. $k_2$

F. Physics of Drift Waves

a. For a uniform plasma, motions with $\nabla \cdot Y_i = 0$ do not perturb the density ($\frac{\partial n_i}{\partial t} = -\nabla \cdot \rho_i = 0$).
b. But, when a density gradient is present,

$$\frac{\partial n_i}{\partial t} = -\nabla \cdot \rho_i / \neq 0 \text{ even when } \nabla \cdot Y_i = 0.$$
c. Here, the $E \times B$ drift pushes plasma of lower density into higher density regions (and vice versa).
d. For the system we examined, this is due to $E_y$.

Since $E_i = -\nabla \phi_i$, it is $-i k_1 \phi_1$ component that leads to these motions. Thus $k_1 \neq 0$ is necessary, otherwise, we just have the usual ion-acoustic waves along the magnetic field.
2. Low Frequency Turbulence in Fusion Devices
   a. Fusion devices (tokamaks, for example) typically have \( \beta \sim 1\% \ll 1 \).
   b. Thus, Alfvén waves travel very close along the mean field.
   c. The low frequency turbulent dynamics in tokamaks is drift wave turbulence due to density gradients in the plasma.
   d. Many studies of turbulence in fusion devices employ the electrostatic approximation as outlined here.

3. DEFINE: Drift Wave Frequency \[ \omega_{\ast} = k_y U_d \]
   a. \[ \omega_{\ast} = \frac{T_e}{2eB_0} k_y (\frac{n_0'}{n_0}) \]
   b. \[ \omega^2 - \omega_{\ast}^2 - k_y^2 c_s^2 = 0 \]
   c. \[ \omega = \omega_{\ast} \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{4k_y^2 c_s^2}{\omega_{\ast}^2}} \right) \]