Lecture #11: Kinetic Theory for Electrostatic Waves in Unmagnetized Plasma

I. Review of Kinetic Theory

A. The Boltzmann Equation (Plasma Kinetic Equation)

1. $$\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \underbrace{\frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})}_{\text{Lorentz Force gives acceleration}} \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{coll}$$

where $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ is the distribution function in 6-D phase space.

2. a. Recall from 029:194 Lecture #11, ratio of collisional to collective effects in a plasma,

$$\frac{\text{collisional effects}}{\text{collective effects}} \sim \frac{\nu_c}{\omega_{pe}} \sim \frac{1}{\underset{\uparrow}{N_D}} \ll 1$$

Number of particles in the Debye sphere.

b. Thus, for many plasmas, we can neglect the effect of collisions,

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = 0$$

Thus, Boltzmann Equation $\Rightarrow$ Vlasov Equation

B. Vlasov-Maxwell System of Equations

1. The starting point for our studies of kinetic theory is this system.

a. Vlasov Equation $$\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0 \quad \text{for } s = i, e$$

b. Maxwell's Equations:

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0} \quad \text{Poisson} \qquad \rho_q = \sum_s \int d^3v\, q_s f_s$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \quad \text{Faraday} \qquad \underset{\sim}{j} = \sum_s \int d^3v\, q_s \underset{\sim}{v} f_s$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t} \quad \text{Ampere-Maxwell}$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

c. This is a closed, integro-differential system of equations for 8 unknowns $f_i(\underset{\sim}{x}, \underset{\sim}{v}, t)$, $f_e(\underset{\sim}{x}, \underset{\sim}{v}, t)$, $\underset{\sim}{E}(\underset{\sim}{x}, t)$, $\underset{\sim}{B}(\underset{\sim}{x}, t)$

I. (Continued)

C. The Distribution Function

1. Maxwellian Distribution

a. For many linear problems, we take the lowest order (equilibrium) distribution function to be Maxwellian.

b. Maxwellian distributions characterize Local Thermodynamic Equilibrium (a maximum entropy state — no free energy.)

c.
$$f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = \frac{n_s(\underset{\sim}{x},t)}{\pi^{3/2} v_{ts}^3(\underset{\sim}{x},t)} e^{-\frac{m_s|\underset{\sim}{v}-\underset{\sim}{U}(\underset{\sim}{x},t)|^2}{2T_s(\underset{\sim}{x},t)}}$$

where DEF: Thermal Velocity $v_{ts}^2 \equiv \dfrac{2T_s(\underset{\sim}{x},t)}{m_s}$

> NOTE: As in Lecture #10, for the rest of the semester, we will absorb Boltzmann's constant $k = 1.38\times10^{-27}\frac{J}{K}$ into the temperature, $kT_s \Rightarrow T_s$, giving temperature T_s in units of energy (J).

d. For steady, uniform conditions (homogeneous in space, $\frac{\partial f_{sm}}{\partial \underset{\sim}{x}}=0$), and no flow velocity $\underset{\sim}{U}=0$, this simplifies to

$$\boxed{f_{sm}(v) = \frac{n_{s0}}{\pi^{3/2} v_{ts}^{3/2}} e^{-\frac{v^2}{v_{ts}^2}}}$$

NOTE: This is a function of only $v = |\underset{\sim}{v}|$.

2. Moments of the distribution function:

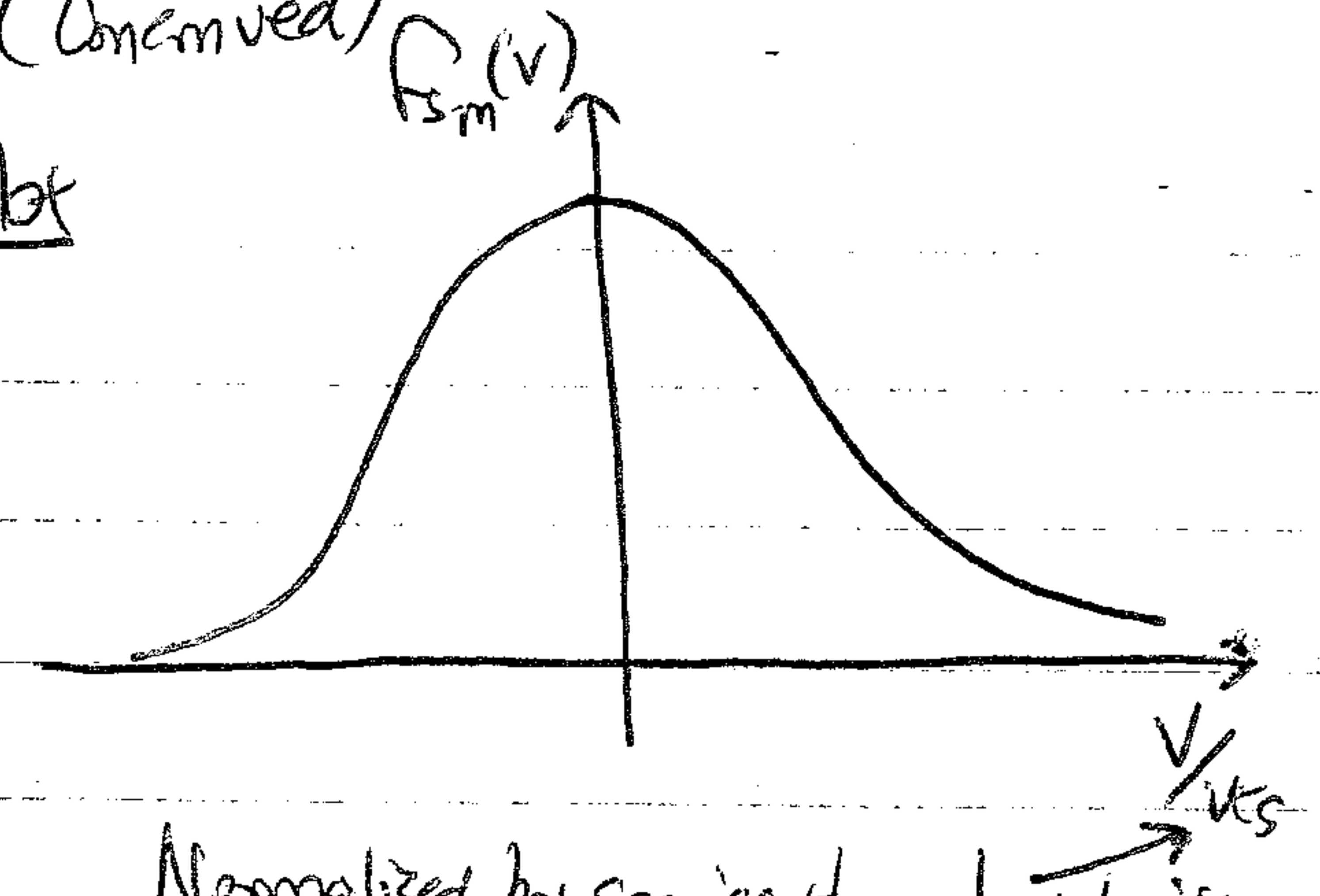
a. Density: $n_{s0} = \int d^3\underset{\sim}{v}\, f_{sm}(v)$

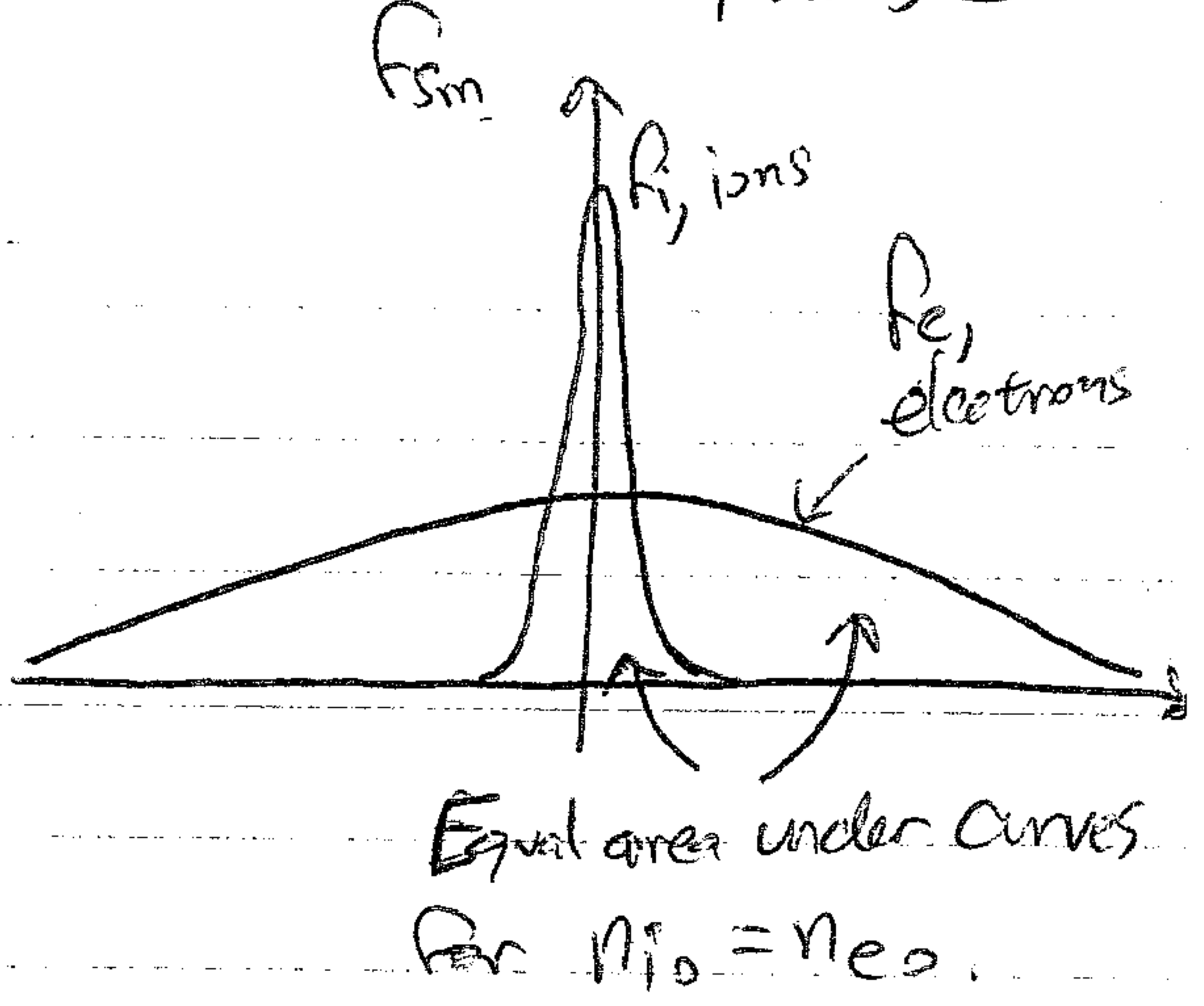
b. Energy: $\frac{3}{2} n_{s0} T_{s0} = \int d^3\underset{\sim}{v}\, \frac{1}{2} m_s v^2 f_{sm}(v)$

Lecture #11 (Continued) Waves (3)

I.C. (Continued)

3. Plot





4. Reduced Distribution Function:

a. DEF: $$F_s(v_z) \equiv \frac{1}{n_{s0}} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \, f_{sm}(v)$$

b. Integrate over two velocity space dimensions v_x & v_y.

c. For Maxwellian,

$$F_s(v_z) = \frac{1}{\cancel{n_{s0}}} \int_{-\infty}^{\infty} d^3\underset{\sim}{v} \left[\frac{\cancel{n_{s0}}}{\pi^{3/2} v_{ts}^{3}} e^{-\frac{(v_x^2+v_y^2+v_z^2)}{v_{ts}^2}} \right]$$

$$= \frac{e^{-\frac{v_z^2}{v_{ts}^2}}}{\pi^{1/2} v_{ts}} \underbrace{\int_{-\infty}^{\infty} \frac{dv_x}{v_{ts}} \frac{e^{-\frac{v_x^2}{v_{ts}^2}}}{\pi^{1/2}}}_{=1} \underbrace{\int_{-\infty}^{\infty} \frac{dv_y}{v_{ts}} \frac{e^{-\frac{v_y^2}{v_{ts}^2}}}{\pi^{1/2}}}_{=1} = \boxed{\frac{e^{-\frac{v_z^2}{v_{ts}^2}}}{\pi^{1/2} v_{ts}} = F_s(v_z)}$$

d. NOTE: $\int_{-\infty}^{\infty} F_s(v_z)\, dv_z = 1$

II. Electrostatic Waves in an Unmagnetized Plasma

A. Setup

1. Electrostatic Approximation, $\underset{\sim}{B}_1 = 0 \Rightarrow \nabla \times \underset{\sim}{E} = 0 \Rightarrow \underset{\sim}{E} = -\nabla\phi$

2. No mean magnetic field, $\underset{\sim}{B}_0 = 0$, ~~Also~~ And $\underset{\sim}{E}_0 = 0$.

3. Vlasov-Maxwell Systems Simplifies to

a. $$\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla\phi \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0 \quad \text{(for ions \& electrons)}$$

b. $$-\nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s \int d^3\underset{\sim}{v}\, q_s f_s$$

Closed system of Equations for $f_e(\underset{\sim}{x}, \underset{\sim}{v}, t)$, $f_i(\underset{\sim}{x}, \underset{\sim}{v}, t)$, $\phi(\underset{\sim}{x}, t)$

II. (Continued)

B. Linearization:

1. NOTE: Now $\underset{\sim}{v}$ is a coordinate and not a variable. Therefore, we don't expand $\underset{\sim}{v}$.

2. Take

a. $$f_s = f_{s0}(\underset{\sim}{v}) + \epsilon f_{s1}(\underset{\sim}{x},\underset{\sim}{v},t)$$

b. $$\phi(\underset{\sim}{x},t) = \cancel{\phi_0}^{\,0} + \epsilon \phi_1(\underset{\sim}{x},t)$$

3. NOTE: a. In steady state, $\frac{\partial f_{s0}}{\partial t} = 0$

b. For a homogeneous plasma, $\frac{\partial f_{s0}}{\partial \underset{\sim}{x}} = 0$.

c. Since $\underset{\sim}{E}_0 = 0 \Rightarrow \phi_0 = 0$.

4. a. $$\epsilon \frac{\partial f_{s1}}{\partial t} + \epsilon \underset{\sim}{v}\cdot\nabla f_{s1} - \epsilon \frac{q_s}{m_s}\nabla\phi_1 \cdot \frac{\partial f_{s0}}{\partial \underset{\sim}{v}} - \epsilon^2 \frac{q_s}{m_s}\nabla\phi_1 \cdot \frac{\partial f_{s1}}{\partial \underset{\sim}{v}} = 0$$

b. $$-\epsilon \nabla^2 \phi_1 = \frac{1}{\epsilon_0}\sum_s \int d^3\underset{\sim}{v}\, q_s f_{s0} + \frac{1}{\epsilon_0}\epsilon \sum_s \int d^3\underset{\sim}{v}\, q_s f_{s1}$$

5. NOTE: $\int d^3\underset{\sim}{v}\, q_s f_{s0} = n_{s0} q_s$, so the first term of RHS of Poisson's equations becomes $\frac{1}{\epsilon_0}\sum_s n_{0s} q_s = 0$ for charge neutral equilibrium.

6. $O(\epsilon)$:

$$\boxed{\begin{aligned} &\frac{\partial f_{s1}}{\partial t} + \underset{\sim}{v}\cdot\nabla f_{s1} - \frac{q_s}{m_s}\nabla\phi_1 \cdot \frac{\partial f_{s0}}{\partial \underset{\sim}{v}} = 0 \\ &-\nabla^2\phi_1 = \frac{1}{\epsilon_0}\sum_s \int d^3\underset{\sim}{v}\, q_s f_{s1} \end{aligned}}$$

C. Fourier Transform in Space and Time

1. As usual, we'll solve this by Fourier transform $\sim e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$

$\nabla \Rightarrow i\underset{\sim}{k}$ $\qquad \frac{\partial}{\partial t} \Rightarrow -i\omega$

2. $$-i\omega f_{s1} + i\underset{\sim}{v}\cdot\underset{\sim}{k} f_{s1} - i\frac{q_s\phi_1}{m_s}\underset{\sim}{k}\cdot\frac{\partial f_{s0}}{\partial \underset{\sim}{v}} = 0$$

$$k^2\phi_1 = \frac{1}{\epsilon_0}\sum_s \int d^3\underset{\sim}{v}\, q_s f_{s1}$$

II. C. (Continued)

3. Solving for f_{s1}:

$$\boxed{f_{s1} = \frac{\frac{-q_s \phi_1}{m_s} \underset{\sim}{k} \cdot \frac{\partial f_{s0}}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}}}$$

4. Substituting f_{s1} into Poisson's Equation:

a. $$k^2 \phi_1 = \frac{1}{\epsilon_0} \sum_s \int d^3\underset{\sim}{v} \left(\frac{-q_s^2 \phi_1}{m_s} \frac{\underset{\sim}{k} \cdot \frac{\partial f_{s0}}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} \right)$$

5. Dividing by k^2 and collecting terms:

$$\underbrace{\left[1 + \sum_s \overbrace{\left(\frac{n_{s0} q_s^2}{\epsilon_0 m_s} \right)}^{\omega_{ps}^2} \frac{1}{k^2 n_{s0}} \int d^3\underset{\sim}{v} \frac{\underset{\sim}{k} \cdot \frac{\partial f_{s0}}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} \right]}_{\text{Dispersion relation } D(\omega, \underset{\sim}{k})} \phi_1 = 0$$

D. Simplifying

1. Take $\underset{\sim}{k} = k\hat{z}$ without loss of generality.

a. $$\underset{\sim}{k} \cdot \frac{\partial f_{s0}}{\partial \underset{\sim}{v}} = k \frac{\partial f_{s0}}{\partial v_z}$$

b. $\underset{\sim}{k} \cdot \underset{\sim}{v} = k v_z$

2.

a. $$D(\omega, \underset{\sim}{k}) = \left\{ 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{k \frac{\partial}{\partial v_z}}{\omega - k v_z} \underbrace{\left[\frac{1}{n_{s0}} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \, f_{s0} \right]}_{= F_{s0}(v_z)} \right\} = 0$$

$= F_{s0}(v_z)$ Reduced Distribution Function

b. $$\boxed{D(\omega, k) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0} / \partial v_z}{v_z - \frac{\omega}{k}} = 0}$$ Dispersion Relation

E. Failure of Fourier Transform Approach

1. The integral above in $D(\omega, k)$ does not converge.

a. At $v_z = \frac{\omega}{k}$, denominator is zero.

b. Unless $\frac{\partial F_{s0}}{\partial v_z}$ (and F_{s0}) $= 0$ at $v_z = \frac{\omega}{k}$, integral does not converge.

2. The failure at $V_z = \frac{\omega}{k}$ occurs when there are particles with velocities that match the phase velocity of the wave.
   $\Rightarrow$ These particles are resonant with the wave.

F. The Bohm-Gross Dispersion Relation

1. Let's consider a plasma of electrons with stationary ions forming a neutralizing background. $n_{i_0} = n_{e_0}$, $f_{i_1} = 0$.
2. Cold Plasma Limit:
   a. Assume electron thermal velocity much less than phase velocity.
   $$V_{te}^2 \simeq \langle v_z^2 \rangle \ll \frac{\omega^2}{k^2}$$
   where $\langle v_z^2 \rangle = \int_{-\infty}^{\infty} dv_z\, v_z^2\, F_{e0}(v_z)$

   b. Strictly, we can only use this approach when $F_{e0}(v_z)\Big|_{v_z = \frac{\omega}{k}} = 0$ (no resonant particles).

3a. Integrate $D(\omega, k)$ by parts

$$\int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{\omega}{k}} = \cancel{\left[\frac{F_{s0}}{v_z - \frac{\omega}{k}}\right]_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} dv_z \frac{F_{s0}}{(v_z - \frac{\omega}{k})^2}$$

$u = \frac{1}{v_z - \frac{\omega}{k}}$ $\quad dv = \frac{\partial F_{s0}}{\partial v_z} dv_z$ $\quad \lim_{v_z \to \pm\infty} F_{s0}(v_z) = 0$

$du = \frac{-dv_z}{(v_z - \frac{\omega}{k})^2}$ $\quad v = F_{s0}$

b. Thus $D(\omega, k) = 1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_{e0}}{(v_z - \frac{\omega}{k})^2} = 0$

4. Expand denominator for $v_z \ll \frac{\omega}{k}$

$$\frac{1}{(v_z - \frac{\omega}{k})^2} = \frac{k^2}{\omega^2 (1 - \frac{k v_z}{\omega})^2} \simeq \frac{k^2}{\omega^2}\left[1 + 2\left(\frac{k v_z}{\omega}\right) + 3\left(\frac{k v_z}{\omega}\right)^2 + \dots\right]$$

5. $D(\omega,\underline{k}) = 1 - \frac{\omega_{pe}^2}{\omega^2}\int_{-\infty}^{\infty} dv_z\, F_{e0}\left[1 + 2\left(\frac{kv_z}{\omega}\right) + 3\left(\frac{kv_z}{\omega}\right)^2\right]$
① ② ③

a. ① $= \int_{-\infty}^{\infty} dv_z\, F_{e0} = 1$

b. ② $= 2\int_{-\infty}^{\infty} dv_z \left(\frac{k}{\omega}\right) v_z\, F_{e0} = 0$ (odd in v_z)

c. ③ $= 3\frac{k^2}{\omega^2}\int_{-\infty}^{\infty} dv_z\, v_z^2\, F_{e0}(v_z) \equiv \frac{3k^2}{\omega^2}\langle v_z^2\rangle$

<u>NOTE:</u> The form of $\langle v_z^2\rangle$ depends on equilibrium distribution function.

6. a. $D(\omega,\underline{k}) = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2\langle v_z^2\rangle}{\omega^2}\right) = 0$

b. $\boxed{\omega^2 = \omega_{pe}^2\left(1 + 3\frac{k^2\langle v_z^2\rangle}{\omega^2}\right)}$

c. <u>NOTE:</u> We have assumed $\frac{k^2\langle v_z^2\rangle}{\omega^2} \ll 1$, so second term is a small correction. This can be solved easily by the <u>method of successive approximations.</u>

7. <u>Method of Successive Approximations:</u>

a. Solve for ω^2 by dropping small term: $\omega^2 = \omega_{pe}^2\left(1 + \cancel{3\frac{k^2\langle v_z^2\rangle}{\omega^2}}\right)$

$\Rightarrow \omega_0^2 = \omega_{pe}^2$

b. Insert first solution ω_0^2 into small term to get second solution ω_1^2:

$\omega_1^2 = \omega_{pe}^2\left(1 + \frac{3k^2\langle v_z^2\rangle}{\underbrace{\omega_0^2}_{=\omega_{pe}^2}}\right) = \omega_{pe}^2 + 3k^2\langle v_z^2\rangle$

c. Thus, we find $\boxed{\omega^2 = \omega_{pe}^2 + 3k^2\langle v_z^2\rangle}$

8. Alternative Explanation of Method of Successive Approximations

a. $\omega^2 = \omega_{pe}^2 \left(1 + \frac{3k^2\langle v_z^2\rangle}{\omega^2}\right)$ Small term

b. Let $x = \omega^2$, $a = \omega_{pe}^2$, $b = 3k^2\langle v_z^2\rangle \Rightarrow x = a\left(1 + \epsilon\frac{b}{x}\right)$

c. $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$

d. $x_0 + \epsilon x_1 = a\left[1 + \epsilon\frac{b}{(x_0 + \epsilon x_1)}\right] \simeq a\left[1 + \frac{\epsilon b}{x_0\left(1 + \frac{\epsilon x_1}{x_0}\right)}\right] \simeq a\left(1 + \frac{\epsilon b}{x_0} - \epsilon^2\frac{b x_1}{x_0^2}\right)$

e. $O(1)$: $x_0 = a \Rightarrow \omega_0^2 = \omega_{pe}^2$

f. $O(\epsilon)$: $x_1 = \frac{ab}{x_0} \Rightarrow \omega_1^2 = \frac{\cancel{\omega_{pe}^2}\, 3k^2\langle v_z^2\rangle}{\cancel{\omega_{pe}^2}} = 3k^2\langle v_z^2\rangle$

g. $x = x_0 + \epsilon x_1 \Rightarrow \boxed{\omega^2 = \omega_{pe}^2 + 3k^2\langle v_z^2\rangle}$

9. Maxwell Equilibrium Distribution:

a. $\langle v_z^2\rangle = \int_{-\infty}^{\infty} dv_z\, v_z^2 F_{e0}(v_z) = v_{te}^2 \int_{-\infty}^{\infty} \frac{dv_z}{v_{te}} \frac{v_z^2}{v_{te}^2} \frac{e^{-\frac{v_z^2}{v_{te}^2}}}{\pi^{1/2}} = v_{te}^2 \int_{-\infty}^{\infty} dy\, y^2 \frac{e^{-y^2}}{\pi^{1/2}}$ $\quad y = \frac{v_z}{v_{te}}$

$= v_{te}^2 \frac{\cancel{\sqrt{\pi}}}{2\cancel{\sqrt{\pi}}} = \left(\frac{\cancel{2}T_e}{m_e}\right)\frac{1}{\cancel{2}} = \frac{T_e}{m_e}$

b. Thus $\omega^2 = \omega_{pe}^2 + 3k^2\frac{T_e}{m_e}$

c. From 029:194 Lect# 24, $\omega^2 = \omega_{pe}^2 + \gamma_e k^2 C_e^2$ for Langmuir Waves

1. DEF: Electron Sound Speed: $C_e^2 \equiv \frac{T_e}{m_e}$

2. In Lecture #24, we took $\gamma_e = 1$ for isothermal conditions, which requires $v_{te} \gg \frac{\omega}{k}$. Here we are in the opposite limit, so we put γ_e back in.

10. Bohm-Gross Dispersion Relation

a. $\omega^2 = \omega_{pe}^2 + \gamma_e k^2 C_e^2$ Warm, Two-Fluid Theory (Adiabatic) (Fluid Closure assumed)

b. $\omega^2 = \omega_{pe}^2 + 3k^2 C_e^2$ Kinetic Theory

c. Results agree for $\gamma_e = 3$ (one degree of freedom, $\gamma = \frac{f+2}{f}$)

~~AA~~ Kinetic Theory gives result without assuming an Equation of State!