I. Laplace-Fourier Solution of Electromagnetic Plasma Waves

A. Setup:
1. Electromagnetic: \( \mathbf{E} = -\nabla \phi \), \( \mathbf{B} = 0 \), \( E_0 = 0 \Rightarrow \phi_0 = 0 \)
2. Vlasov-Maxwell System:
\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s - \frac{q_s}{m_s} \mathbf{E} \cdot \nabla \phi - \frac{q_s}{m_s} \frac{\partial f_s}{\partial \mathbf{v}} = 0
\]
\[
-\nabla^2 \phi = \frac{1}{\varepsilon_0} \sum_{\mathbf{k}} \int d^3 \mathbf{v} q_s f_s
\]
3. Take \( k = k' \sqrt{2} \)

B. Linearization 1. \( \phi = f_s (\mathbf{v}) + \epsilon f_1 (\mathbf{x}, \mathbf{v}, t) \)
\[
\phi = \phi_0 + \epsilon \phi_1 (\mathbf{x}, \mathbf{v}, t)
\]
2. \( A + O(\epsilon) \):
\[
\begin{align*}
\frac{\partial f_{s1}}{\partial t} + \mathbf{v} \cdot \nabla f_{s1} - \frac{q_s}{m_s} \mathbf{E} \cdot \nabla \phi & = 0 \\
-\nabla^2 \phi & = \frac{1}{\varepsilon_0} \sum_{\mathbf{k}} \int d^3 \mathbf{v} q_s f_s
\end{align*}
\]

C. Fourier Transform in Space Only \( \mathbf{D} \Rightarrow \mathbf{i} \mathbf{k} \)
1. \( \frac{\partial f_{s1}}{\partial t} + i \mathbf{v} \cdot \mathbf{k} f_{s1} - i \frac{q_s}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} = 0 \)
2. \( \mathbf{k}^2 \phi_1 = \frac{1}{\varepsilon_0} \sum_{\mathbf{k}} \int d^3 \mathbf{v} q_s f_s \)

D. Laplace Transform in Time: \( \tilde{f}_s (p) = \int_0^{\infty} dt f_s (t) e^{-pt} \)
1. \( \tilde{f}_s (p) + i \mathbf{v} \cdot \mathbf{k} \tilde{f}_s (p) - i \frac{q_s}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} = 0 \)
2. Using \( \tilde{f}' (p) = pf (p) - f (0) \), we get
\[
(p + i \mathbf{v} \cdot \mathbf{k}) \tilde{f}_s (p) = i \frac{q_s}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} + f (0)
\]
Lecture 13 (Continued)

2. Solving for $f_s(p)$

$$
\hat{f}_s(p) = \frac{i \kappa \frac{\partial \phi}{\partial x} q_s \Phi(p)}{p + i k \cdot x} + \frac{f_s(0)}{p + i k \cdot x}
$$

The poles in this solution are due to $\Phi(p)$ poles and $p = -i k \cdot x$

E. Substitute $\hat{f}_s(p)$ into Bisson's Equation to solve for $\tilde{\phi}_1(p)

1. $k^2 \tilde{\phi}_1 = \frac{1}{\epsilon_0} \frac{\partial}{\partial x} \int \frac{q_s \Phi(0)}{p + i k \cdot x} + \frac{f_s(0)}{p + i k \cdot x}$

Note: $\Phi(p)$ does not depend on $y$.

2. Divide by $k^2$ and collect $\tilde{\phi}_1(p)$ terms:

a. $\tilde{\phi}_1 \left[ 1 - \frac{1}{k^2} \frac{q_s \Phi(0)}{\epsilon_0 \omega_m} \right] = \frac{1}{k^2 \epsilon_0} \frac{\partial}{\partial x} \int \frac{q_s \Phi(0)}{p + i k \cdot x} + \frac{f_s(0)}{p + i k \cdot x}$

Dispersion Relation

$D(p, k)$

Initial Conditions

$N(p, k)$

b. Solution $\tilde{\phi}_1 \cdot D(p, k) = 0$ gives normal modes of the system.

c. Thus

$$
\tilde{\phi}_1(p) = \frac{N(p, k)}{D(p, k)}
$$

d. Inverse Laplace Transform $\tilde{\phi}_1(p)$ (by Residue Theorem is due to poles in $N(p, k)$ and zeros at $D(p, k)$)

F. Simplify using $k = k_z$ and Reduced Distribution Function $F_{so}(V_e)$

1. $F_{so}(V_e) = \frac{1}{N^0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{so}(y)$
Lecture #13 (Continued)
2. Thus a. \( D(p, k) = 1 - \frac{\xi}{s k^2} \int_{-\infty}^{\infty} \frac{dvz}{vz - \frac{ip}{k}} \)

\[ = 1 - \frac{\xi}{s k^2} \int_{-\infty}^{\infty} \frac{dvz}{vz - \frac{ip}{k}} \]

b. Similarly

\[ Nt(p, k) = \frac{i q s n_0}{s k^3} \int_{-\infty}^{\infty} \frac{F_3(0)}{vz - \frac{ip}{k}} \]

3. Our solution \( \tilde{\phi}_1(p, k) \) is then given by

\[ \tilde{\phi}_1(p, k) = \frac{-i q s n_0}{s k^3} \int_{-\infty}^{\infty} \frac{F_3(0)}{vz - \frac{ip}{k}} \]

4. We want to find

\[ \phi(k, t) = \frac{1}{2\pi i} \int_{C_R} \tilde{\phi}(p, k) e^{pt} dp \]

Using the Residue Theorem.

G. Evaluation of \( \phi(k, t) \) Using Residue Theorem

1. To evaluate \( \phi(k, t) \) using the Residue Theorem, we choose the contour by completing the loop at \( \text{Re}(p) \to -\infty \) (This is section \( C_R \))

Thus

\[ \int_C \tilde{\phi}(p, k) e^{pt} dp = \int_{C_R} \tilde{\phi}(p, k) e^{pt} dp + \int_{\text{other part of contour}} \tilde{\phi}(p, k) e^{pt} dp \]
To evaluate contour integral using the Residue Theorem requires that \( f(z) \) be analytic within and on contour \( C \).

But, the function \( f(ke^{i\alpha}) \) was only defined for \( \Re(p) > 0 \).

\[ \Rightarrow \text{Thus we must analytically continue } f(ke^{i\alpha}) \text{ to the negative real half plane } \Re(p) < 0. \]

c. This is not straightforward due to the \( V_2 \)-integral in both \( D(p, k) \) and \( N(p, k) \). For example,

\[ D(p, k) = 1 - \frac{\alpha k^2}{\pi} \int_{-\infty}^{\infty} \frac{df_{\text{sc}}/dv}{V_2 - \frac{i\alpha}{k}}. \]

d. This function is discontinuous on the line \( \Re(p) = 0 \).

Why? Remember \( p = \sigma - i\alpha \), so the denominator is

\[ V_2 - \frac{1}{k}(\sigma - i\alpha) = V_2 - \frac{\alpha}{k} - \frac{i\alpha}{k}. \]

\[ \Rightarrow \text{If } \Re(p) = \sigma = 0, \text{ then we have } \int_{-\infty}^{\infty} \frac{df_{\text{sc}}/dv}{V_2 - \frac{\alpha}{k}} \]

and the integral becomes undefined or \( V_2 = \frac{\alpha}{k} \).

e. Since we must perform the contour integral over the entire complex plane \( p \), this problem of \( \Re(p) = 0 \) must be resolved.

H. Landau's Analytic Continuation of \( D(p, k) \) and \( N(p, k) \)

1. Landau solved this problem by carrying out a careful analytic continuation of \( D(p, k) \) and \( N(p, k) \) to \( \Re(p) < 0 \).
2. Consider the case \( k > 0 \) (\( k < 0 \) is analogous). The pole at \( v_z = \frac{1}{k} \) then lies at the following points in complex \( v_z \) space:

\[
\begin{align*}
\text{Re}(p) > 0 & \quad \text{Re}(p) = 0 & \quad \text{Re}(p) < 0
\end{align*}
\]

a. Treating the integral \( \int_{-\infty}^{\infty} du \) as a contour integral in complex \( v_z \) space, Landau observed the contour of integration so that it always passed below the pole in \( v_z \) space.

b. In this way, the functions \( \Phi(p, k) \) and \( V(p, k) \) [and thus \( f(p, k) \)] are analytically continued into the \( \text{Re}(p) < 0 \) half of the complex \( p \) plane.

c. Now we can go ahead and use the Residue Theorem to evaluate \( \int_{C} \Phi(p, k) e^{pt} dp \).

3. We'll look at concrete examples of this \( v_z \) integration soon.

b. For Maxwellian equilibrium distribution, this gives rise to the Plasma Dispersion Function.

4. Evaluation of \( \Phi(k, t) \)

1. Reminder: \( f(t) = \frac{1}{2\pi i} \int_{C} \Phi(p) e^{pt} dp \)
Lecture 13 (Continued)
1. I. (Continued)

2. Let \( \tilde{\phi}(k,p) \) be the function defined by

\[
\int_{C_p} \tilde{\phi}(k,p)e^{pt} \, dp = \int_{-\infty}^{0-10^6} \tilde{\phi}(k,p)e^{pt} \, dp + \int_{0+10^6}^{\infty} \tilde{\phi}(k,p)e^{pt} \, dp = 2\pi i \phi(k,t) \quad \text{as} \quad \text{Re}(p) \to -\infty
\]

b. By deformation of paths, we can see that the only contribution to integral is due to residues. Deform \( C \) to \( \text{Re}(p) \to -\infty \), except if hangs up on poles:

\[
\text{As} \ \text{Re}(p) \to \infty, \quad \text{These sections cancel}
\]

Contribution due to residues or poles,

3. Thus, we find

\[
\phi(k,t) = \sum_{p \in \mathcal{R}} \text{Res} \left[ \tilde{\phi}(k,p) e^{pt} \right]
\]

4. Remember, \( p \)'s are complex, \( p = \delta - iw \), so solutions typically have a behavior, \( e^{\delta t} e^{-iw t} \), oscillatory with frequency \( w \) and a growth rate for \( \delta > 0 \), or damping rate for \( \delta < 0 \).
II. Solution for Cauchy Velocity Distribution

A. Cauchy Velocity Distribution
1. A simple analytical distribution function is
   \[ F_0(c, v_x) = \frac{c}{\pi} \left( \frac{1}{c^2 + v_x^2} \right) \]
2. \text{NOTE:} \ \int_{-\infty}^{\infty} F_0(c, v_x) \, dv_x = 1
3. Consider ions immobile, so \( F_{0i} = F_{0e} \) and \( A_1 = 0 \).

B. Velocity Integral over \( v_x \)
1. Our Dispersion Relation is
   \[ D(c, k) = 1 - \frac{\alpha p^2}{k^2} \int_{-\infty}^{\infty} \frac{F_0}{(v_x - \frac{ik}{c})^2} \, dv_x \]
   where we only consider the electron contribution since ions are immobile.
2. We can integrate by parts (as done in lect 4 ll, II. F. 3.) to yield
   \[ D(c, k) = 1 - \frac{\alpha p^2}{k^2} \int_{-\infty}^{\infty} \frac{F_0}{(v_x - \frac{ik}{c})^2} \, dv_x = 1 - \frac{\alpha p^2}{k^2} \int_{-\infty}^{\infty} \frac{1}{(v_x - \frac{ik}{c})(v_x + \frac{ik}{c})(v_x - \frac{ik}{c})^2} \, dv_x \]
3. For \( k > 0 \):
   \[ C \quad \text{Cc} \quad + \frac{ik}{c} \quad - \frac{ik}{c} \quad \text{Cc} \quad C \]
   \[ \begin{array}{c}
   \text{a. Close to } \text{Im}(v_x) \to -\infty \\
   \text{b. Let } g(v_x) = \frac{1}{(v_x - \frac{ik}{c})(v_x + \frac{ik}{c})(v_x - \frac{ik}{c})^2} \\
   \text{c. Thus } g(v_x) = g(v_x) + g(v_x) \\
   \text{d. Thus for pole at } v_x = -\frac{ik}{c} \\
   \text{e. So we find for } k > 0 : \\
   D(c, k) = 1 + \frac{\alpha p^2}{k^2} \frac{1}{\text{Res}[g(v_x)]} \]
Lesson 13 (Continued)

II. B. (Continued)

4. Similarly for \( k < 0 \)
   a. Close in upper half plane \( \text{Im}(V_2) \to \infty \) (CCW orientation).
   b. Thus \( \sum_{\infty}^{0} \Sigma \text{dv} \Sigma g(V_2) = 2 \pi i \epsilon \leq \text{Res} [g(V_2)] \rightarrow \text{pole or } V_2 = t \epsilon \)
   \[
   = 2 \pi i \frac{1}{2i \epsilon (i \epsilon - \varphi)^2} = \frac{\pi \epsilon}{C (C - \varphi^2)}
   \]
   c. Thus \( D(p, k) = 1 + \frac{a_p^2}{(p - k C)^2} \)

5. Noting that for \( k > 0 \), \( k = k_1 \) and for \( k < 0 \) \( k = -1k_1 \), we can write these as a single equation
   \[
   D(p, k) = 1 + \frac{a_p^2}{(p + k_1 C)^2} = 0
   \]
   a. \textbf{NOTE:} Since this solution is a polynomial, analytic continuation to the \( \text{Re}(p) < 0 \) plane is trivial.

6. Roots of dispersion relation are
   \[
   p = -k_1 C \pm i a_p
   \]

C. Solving for \( N(k, p) \)

1. \( N(k, p) = -i \frac{\left| a_p \right|}{s C_0 k^2} \sum_{0}^{\infty} \text{dv} \epsilon \frac{F_5(k_2, o, o)}{V_2 + 1/k}
   
   a. If we have a specific form for the initial conditions \( F_5(k, x, o) \), then we can perform the integral analogous to the procedure above.

b. An important point is that, as long as \( F_5(k, x, o) \) do not have any singularities or discontinuities, the result of the integration will not have any singularities. \( \Rightarrow \) thus, no poles in \( N(k, p) \)
II. C. (Continued)

2. Rather than solve for a specific form of \( F(x, y, z) \), we note

\[\phi(k, p) D(k, p) = N(k, p) \quad \text{(Dispersive Relation)} \]

\[\phi(k, p) = \phi(k, 0) = \phi(k, 0) \quad \text{(Initial Conditions)}\]

a. We simply denote \( N(k, p) = \phi(k, 0) \) since it is determined by the initial conditions.

b. Thus \( \phi(k, p) = \frac{\phi(k, 0)}{1 + \frac{cp^2}{(p+ikc)^2}} = \frac{\phi(k, 0)}{(p+ikc)^2 + cp^2} \).

D. Completing Solution for \( \phi(k, t) \)

1. As we solved earlier (I. I. 3.), \( \phi(k, t) = \sum_{j=1}^{\infty} \frac{\phi(k, 0)}{p_j} e^{p_j t} \).

a. Here \( \phi(k, p) \cdot e^{pt} = \frac{(p+ikc)^2 \phi(k, 0) e^{pt}}{(p+ikc-icp)(p+ikc+icp)} \).

Poles are roots: \( p = -kC + icp \) & \( p = -kC - icp \).

2. Thus \( \phi(k, t) = \frac{(ikC+icp - ikC)^2 \phi(k, 0) e^{-ikC - icpt}}{(-ikC+icp + ikC - icp)(-ikC+icp + ikC + icp)} \) + \( \frac{(ikC-icp + ikC)^2 \phi(k, 0) e^{-ikC + icpt}}{(-ikC+icp + ikC - icp)(-ikC-icp + ikC - icp)} \).

\( \phi(k, t) = -\frac{cp^2 \phi(k, 0)}{2icp^2} e^{-ikC - icpt} + \frac{cp^2 \phi(k, 0)}{2icp^2} e^{-ikC + icpt} \).

\( \phi(k, t) = -\phi(k, 0) e^{-i(kC - icpt)} \left( \frac{e^{icpt} - e^{-icpt}}{2i} \right) \).