Lecture #13 Landau Damping of Electrostatic Waves

I. Laplace-Fourier Solution of Electrostatic Plasma Waves

A. Setup:

1. Electrostatic: $\underset{\sim}{E} = -\nabla\phi$, $\underset{\sim}{B}=0$, $E_0 = 0 \Rightarrow \phi_0 = 0$
2. Vlasov-Maxwell System:

$$\frac{\partial f_s}{\partial t} + \underset{\sim}{v}\cdot\nabla f_s - \frac{q_s}{m_s}\nabla\phi\cdot\frac{\partial f_s}{\partial \underset{\sim}{v}} = 0$$

$$-\nabla^2\phi = \frac{1}{\epsilon_0}\sum_s \int d^3\underset{\sim}{v}\, q_s f_s$$

3. Take $\underset{\sim}{k} = k\hat{\underset{\sim}{z}}$

B. Linearization

1.
$$f_s = f_{s_0}(\underset{\sim}{v}) + \epsilon f_{s_1}(\underset{\sim}{x},\underset{\sim}{v},t)$$
$$\phi = \cancel{\phi_0}^{\,0} + \epsilon\,\phi_1(\underset{\sim}{x},t)$$

2. At $\mathcal{O}(\epsilon)$: a.
$$\frac{\partial f_{s_1}}{\partial t} + \underset{\sim}{v}\cdot\nabla f_{s_1} - \frac{q_s}{m_s}\nabla\phi_1\cdot\frac{\partial f_{s_0}}{\partial \underset{\sim}{v}} = 0$$
b.
$$-\nabla^2\phi_1 = \frac{1}{\epsilon_0}\sum_s\int d^3\underset{\sim}{v}\, q_s f_{s_1}$$

C. Fourier Transform in Space Only $\nabla \Rightarrow i\underset{\sim}{k}$

1. a.
$$\frac{\partial f_{s_1}}{\partial t} + i\,\underset{\sim}{v}\cdot\underset{\sim}{k} f_{s_1} - i\frac{q_s\phi_1}{m_s}\underset{\sim}{k}\cdot\frac{\partial f_{s_0}}{\partial\underset{\sim}{v}} = 0$$
b.
$$k^2\phi_1 = \frac{1}{\epsilon_0}\sum_s\int d^3\underset{\sim}{v}\, q_s f_{s_1}$$

D. Laplace Transform in Time: $\tilde{f}_s(p) = \int_0^\infty dt\, f_{s_1}(t)\, e^{-pt}$

1. a.
$$\tilde{f}_s'(p) + i\underset{\sim}{v}\cdot\underset{\sim}{k}\,\tilde{f}_s(p) - i\frac{q_s\tilde{\phi}_1(p)}{m_s}\underset{\sim}{k}\cdot\frac{\partial f_{s_0}}{\partial\underset{\sim}{v}} = 0$$

b. Using $\tilde{f}'(p) = p\tilde{f}(p) - f(0)$, we get

$$\boxed{(p + i\underset{\sim}{v}\cdot\underset{\sim}{k})\tilde{f}_s(p) = i\frac{q_s\tilde{\phi}_1(p)}{m_s}\underset{\sim}{k}\cdot\frac{\partial f_{s_0}}{\partial\underset{\sim}{v}} + f(0)}$$

I. D. (Continued)

2. Solving for $\tilde{f}_s(p)$

$$\boxed{\tilde{f}_s(p) = \frac{i\,\underset{\sim}{k}\cdot\frac{\partial f_{s0}}{\partial \underset{\sim}{v}}\,\frac{q_s\tilde{\phi}_1(p)}{m_s}}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}} + \frac{f_s(0)}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}}}$$

The poles in this solution are due to $\tilde{\phi}_1(p)$ poles and $p = -i\underset{\sim}{k}\cdot\underset{\sim}{v}$

E. Substitute $\tilde{f}_s(p)$ into Poisson's Equation to Solve for $\tilde{\phi}_1(p)$

1. $$k^2\tilde{\phi}_1 = \frac{1}{\epsilon_0}\sum_s \int d^3\underset{\sim}{v}\, q_s \left\{ \frac{i\underset{\sim}{k}\cdot\frac{\partial f_{s0}}{\partial\underset{\sim}{v}}\frac{q_s\tilde{\phi}_1(p)}{m_s}}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}} + \frac{f_s(0)}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}} \right\}$$

NOTE: $\tilde{\phi}_1(p)$ does not depend on $\underset{\sim}{v}$.

2. Divide by k^2 and collect $\tilde{\phi}_1(p)$ terms:

a. $$\tilde{\phi}_1 \underbrace{\left[1 - \sum_s \frac{\left(\frac{q_s^2 n_0}{\epsilon_0 m_e}\right)}{k^2 n_0}\int d^3\underset{\sim}{v}\,\frac{i\underset{\sim}{k}\cdot\frac{\partial f_{s0}}{\partial\underset{\sim}{v}}}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}}\right]}_{\substack{\text{Dispersion Relation} \\ D(p,\underset{\sim}{k})}} = \underbrace{\frac{1}{k^2\epsilon_0}\sum_s\int d^3\underset{\sim}{v}\,\frac{q_s f_s(0)}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}}}_{\substack{\text{Initial Conditions} \\ N(p,\underset{\sim}{k})}}$$

b. Solution to $D(p,\underset{\sim}{k}) = 0$ gives normal modes of the system.

c. Thus $$\boxed{\tilde{\phi}_1(p) = \frac{N(p,\underset{\sim}{k})}{D(p,\underset{\sim}{k})}}$$

d. Inverse Laplace Transform of $\tilde{\phi}_1(p)$ by Residue Theorem is due to poles in $N(p,\underset{\sim}{k})$ and zeros of $D(p,\underset{\sim}{k})$

F. Simplify Using $\underset{\sim}{k} = k\hat{\underset{\sim}{z}}$ and Reduced Distribution Function $F_{s0}(v_z)$

1. $$F_{s0}(v_z) \equiv \frac{1}{n_0}\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\, f_{s0}(\underset{\sim}{v})$$

Lecture #13 (Continued)

I. F. (Continued)

2. Thus a. $D(p,\underset{\sim}{k}) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \underbrace{\frac{ik\frac{\partial F_{s0}}{\partial v_z}}{p+ikv_z}}_{=\frac{\frac{\partial F_{s0}}{\partial v_z}}{v_z - \frac{ip}{k}}} = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\frac{\partial F_{s0}}{\partial v_z}}{v_z - \frac{ip}{k}}$

b. Similarly

$$N(p,\underset{\sim}{k}) = \sum_s \frac{-iq_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(0)}{v_z - \frac{ip}{k}}$$

3. Our Solution $\tilde{\phi}_1(\underset{\sim}{k},p)$ is then given by

$$\boxed{\tilde{\phi}_1(p,\underset{\sim}{k}) = \frac{-i\sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(0)}{v_z - ip/k}}{1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - ip/k}}}$$

Pieces of Solution due to:

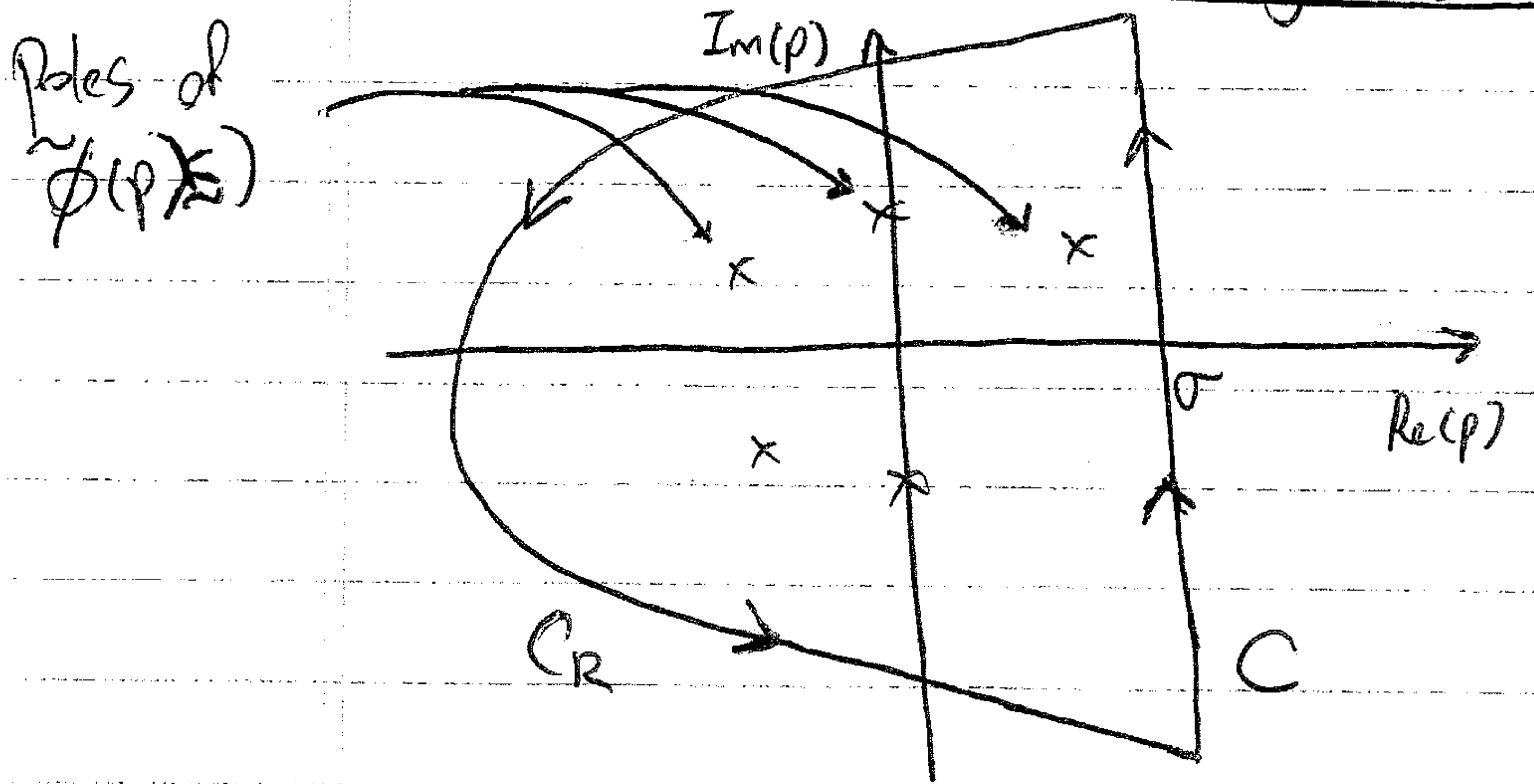
Poles in Numerator

Zeros in Denominator $D(p,\underset{\sim}{k}) = 0$ Normal Modes!

4. We want to find $\phi(\underset{\sim}{k},t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{\phi}(\underset{\sim}{k},p) e^{pt}$

Using the Residue Theorem.

G. Evaluation of $\phi(\underset{\sim}{k},t)$ Using Residue Theorem



1. To Evaluate $\phi(\underset{\sim}{k},t)$ using the Residue Theorem, we close the contour by completing the loop at $Re(p) \to -\infty$ (This is section C_R)

Thus $\int_C dp\, \tilde{\phi}(\underset{\sim}{k},p) e^{pt} = \int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{\phi}(\underset{\sim}{k},p) e^{pt} + \int_{C_R} dp\, \tilde{\phi}(\underset{\sim}{k},p) e^{pt}$

Lecture #13 (Continued)

I. G. (Continued)

2. a. To evaluate contour integral using the Residue Theorem requires that $\tilde{\phi}(k,p)$ be analytic within and on contour C.

b. But, the function $\tilde{\phi}(k,p)$ was only defined for $Re(p)>0$.

$$\boxed{\Rightarrow \text{Thus we must analytically continue } \tilde{\phi}(k,p) \text{ to the negative Real half plane } Re(p)<0.}$$

c. This is not straightforward due to the v_z-integral in both $D(p,k)$ and $N(p,k)$. For example,

$$D(p,k) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \, \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{ip}{k}}$$

d. This function is discontinuous on the line $Re(p)=0$.

Why? ① Remember $p = \gamma - i\omega$, so the denominator is

$$v_z - \frac{i}{k}(\gamma - i\omega) = v_z - \frac{\omega}{k} - \frac{i\gamma}{k}$$

③ If $Re(p) = \gamma = 0$, then we have $\displaystyle\int_{-\infty}^{\infty} dv_z \, \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{\omega}{k}}$

and the integral becomes undefined at $v_z = \frac{\omega}{k}$.

e. Since we must perform our contour integral over the entire complex plane p, this problem at $Re(p)=0$ must be resolved.

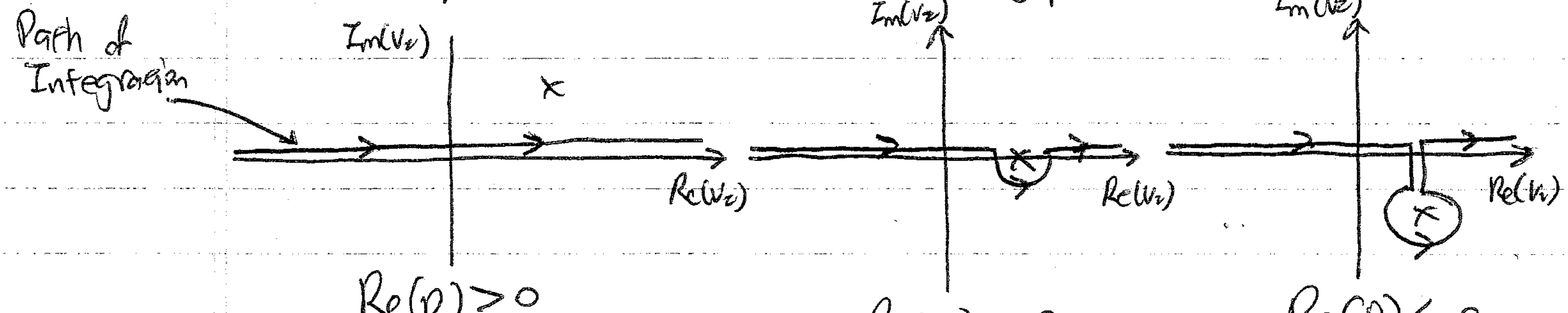
H. Landau's Analytic Continuation of $D(p,k)$ and $N(p,k)$

1. Landau solved this problem by carrying out a careful analytic continuation of $D(p,k)$ and $N(p,k)$ to $Re(p)<0$.

Lecture #13 (Continued) Howes

I. H. (Continued)

2. Consider the case $k>0$ ($k<0$ is analogous). The pole at $v_z = \frac{ip}{k}$ then lies at the following points in complex v_z space.



Re(p) > 0 Re(p) = 0 Re(p) < 0

a. Treating the integral $\int_{-\infty}^{\infty} dv_z$ as a contour integration in complex v_z space, Landau distorted the contour of integration so that it <u>always passes below the pole</u> in v_z space

b. In this way, the functions $D(p,\underline{k})$ and $N(p,\underline{k})$ [and thus $\tilde{\phi}(p,\underline{k})$] are analytically continued into the Re(p)<0 half of the complex p plane.

c. Now we can go ahead and use the Residue Theorem to evaluate $\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{\phi}(p,\underline{k}) e^{pt}$.

3. a. We'll look at concrete examples of this v_z integration soon.

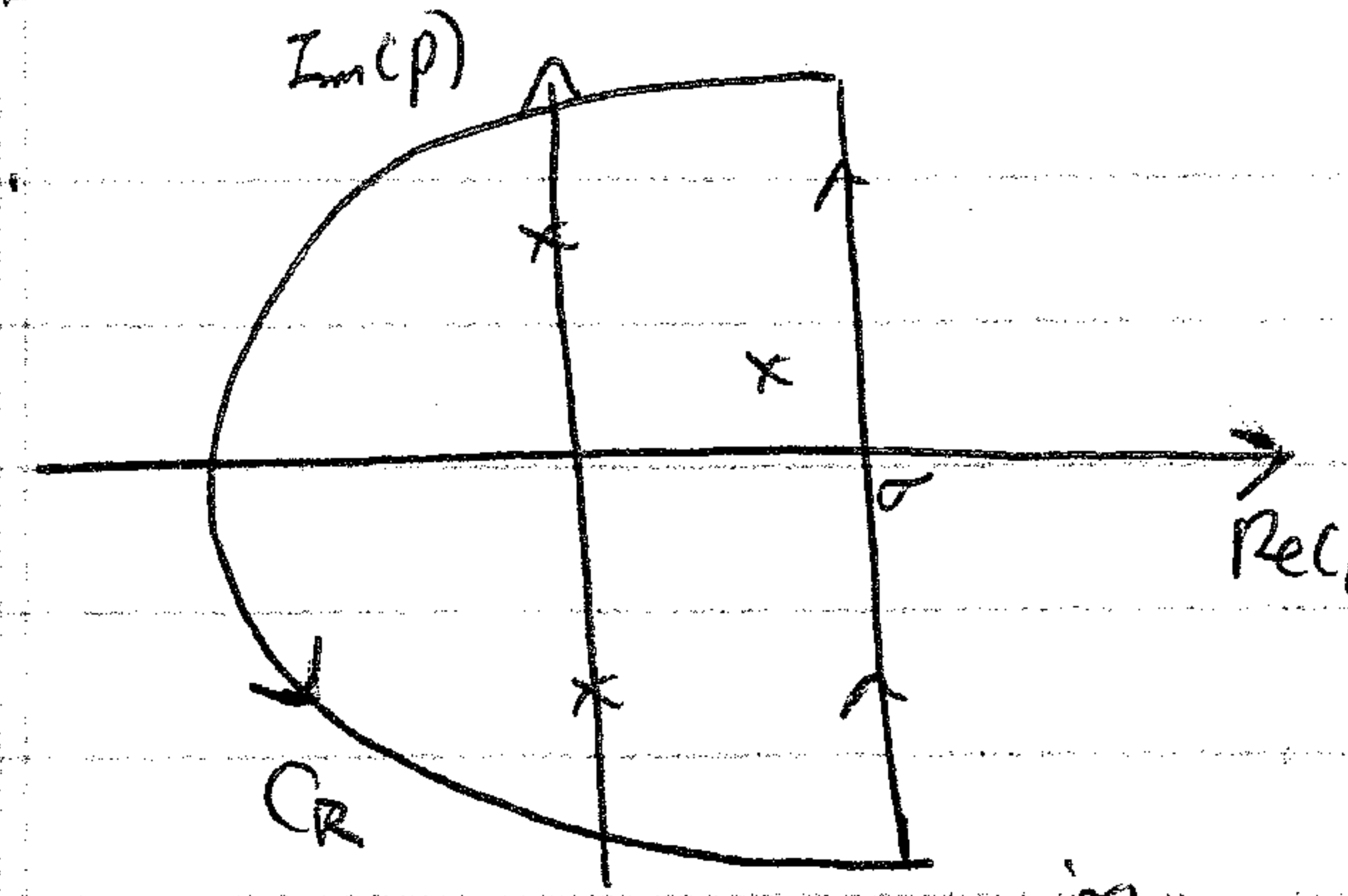
b. For Maxwellian equilibrium distribution, this gives rise to the <u>Plasma Dispersion Function</u>.

I. Evaluation of $\phi(\underline{k},t)$

1. Remember $f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{f}(p) e^{pt}$

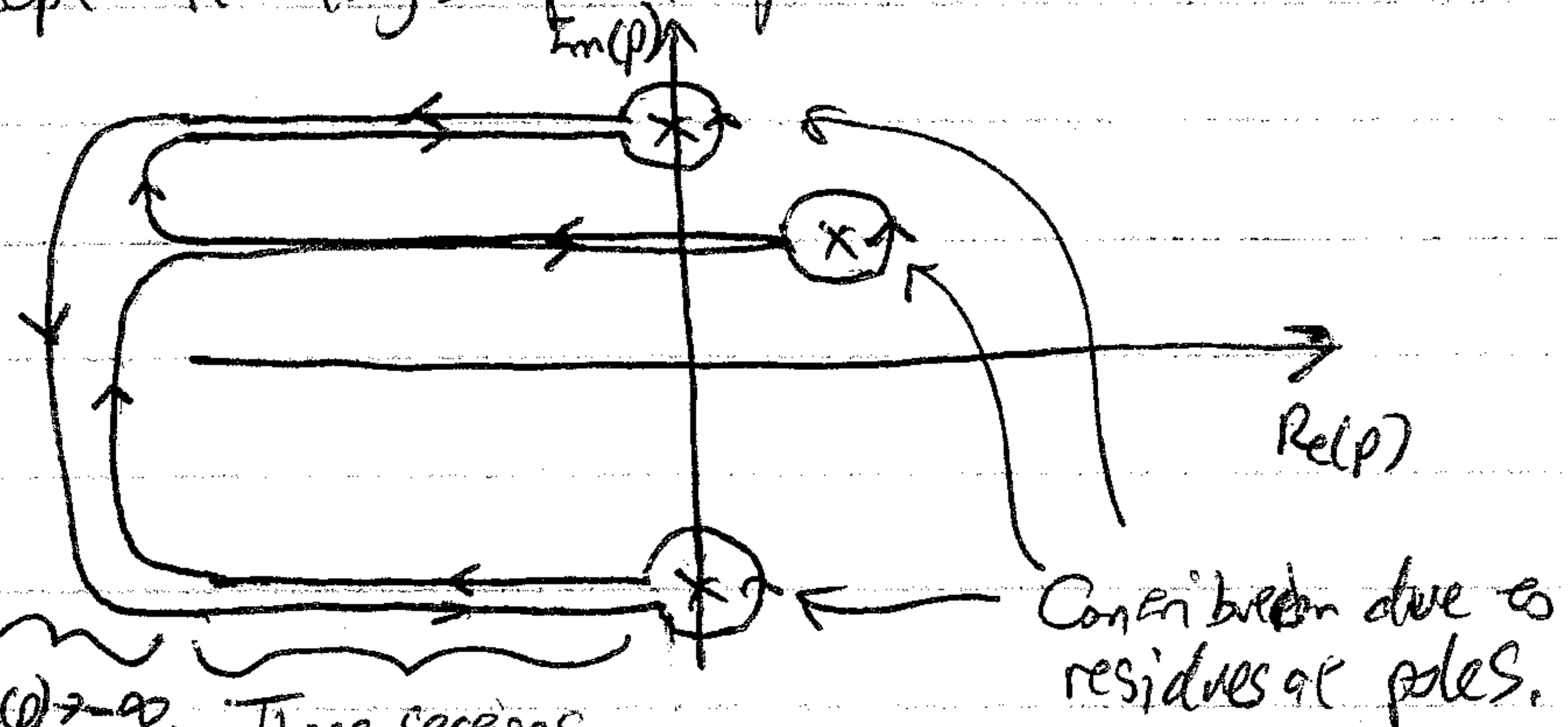
I. I. (Continued)

2.



a. $$\oint_C dp\, \tilde{\phi}(\underline{k},p)e^{pt} = \underbrace{\int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{\phi}(\underline{k},p)e^{pt}}_{=2\pi i\, \phi(\underline{k},t)} + \underbrace{\cancel{\int_{C_R} dp\, \tilde{\phi}(\underline{k},p)e^{pt}}}_{\text{As } Re(p)\to-\infty} = 2\pi i \sum_j \underset{p=p_j}{Res}\left[\tilde{\phi}(p)e^{pt}\right]$$

b. By <u>deformation of paths</u>, we can see that the only contribution to integral is due to residues. Deform C to $Re(p)\to-\infty$, except it hangs up at poles:



3. Thus, we find $$\boxed{\phi(\underline{k},t) = \sum_j \underset{p=p_j}{Res}\left[\tilde{\phi}(\underline{k},p)\, e^{pt}\right]}$$

4. Remember, p's are complex, $p=\gamma-i\omega$, so solutions typically have a behavior, $\sim e^{\gamma t} e^{-i\omega t}$, oscillatory with frequency ω and a growth rate for $\gamma>0$, or damping rate for $\gamma<0$.

II. Solution for Cauchy Velocity Distribution

A. Cauchy Velocity Distribution

1. A simple analytical distribution function is

DEF: Cauchy Reduced Velocity Distribution $F_{0_C}(v_z) = \frac{c}{\pi}\left(\frac{1}{c^2+v_z^2}\right)$

   a. **NOTE:** $\int_{-\infty}^{\infty} dv_z \, F_{0_C}(v_z) = 1$

2. Consider ions immobile, so $F_{0i} = F_{0e}$ and $f_1 = 0$.

B. Velocity Integral over v_z

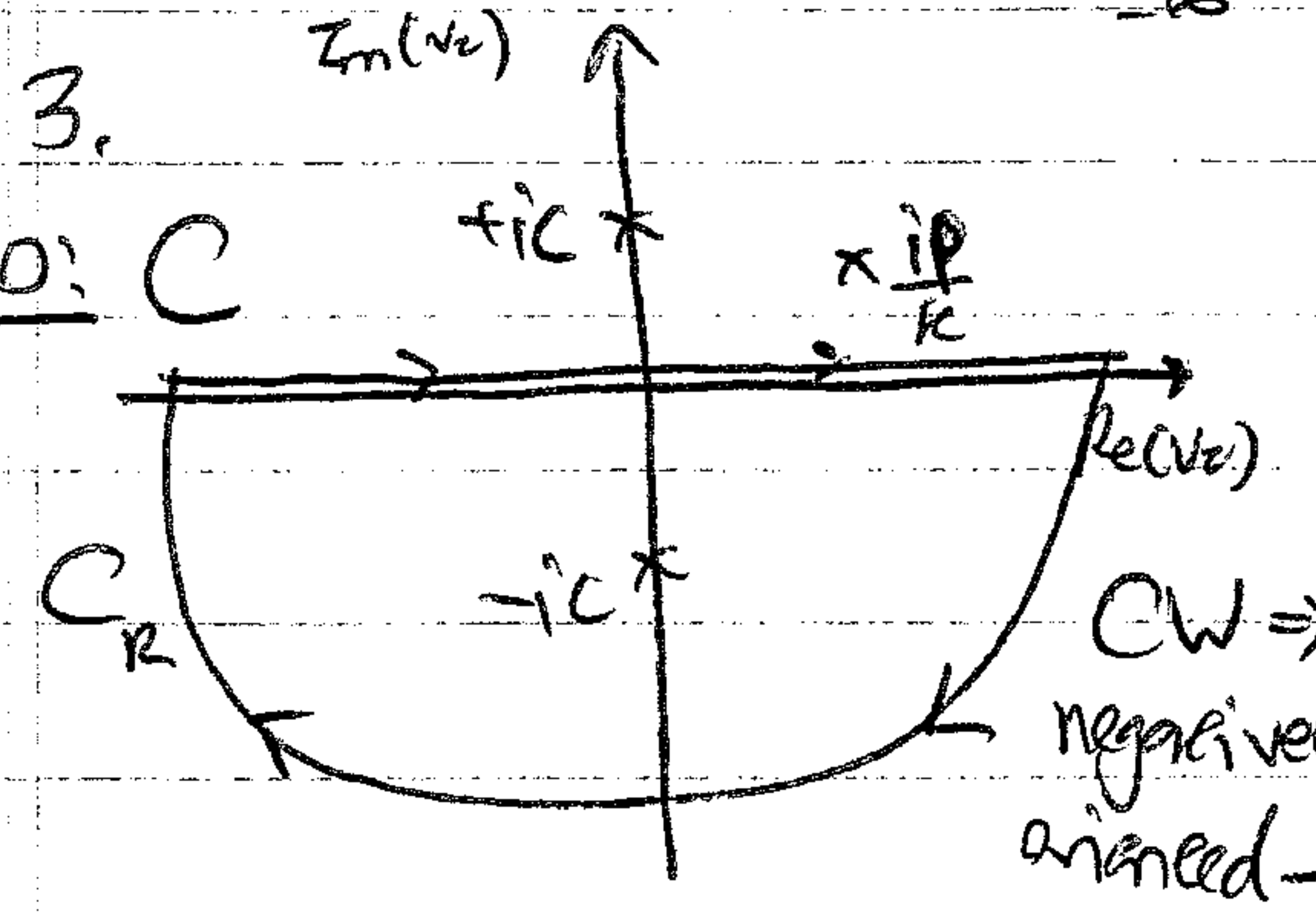
1. Our Dispersion Relation is $D(p,\underline{k}) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{v_z - \frac{ip}{k}}$

where we only consider the electron contribution since ions are immobile.

2. We can integrate by parts (as done in lect #11, II. F. 3.) to yield

$$D(p,\underline{k}) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{F_0}{(v_z - \frac{ip}{k})^2} = 1 - \frac{\omega_p^2}{k^2}\frac{c}{\pi}\int_{-\infty}^{\infty} dv_z \frac{1}{(v_z - ic)(v_z+ic)(v_z - \frac{ip}{k})^2}$$

3. For $k>0$:



   a. Close at $\mathrm{Im}(v_z) \to -\infty$

   b. Let $g(v_z) = \frac{1}{(v_z - ic)(v_z+ic)(v_z-\frac{ip}{k})^2}$

   c. Thus $\int_C dv_z g(v_z) = \int_{-\infty}^{\infty} dv_z g(v_z) + \cancel{\int_{C_R} dv_z g(v_z)} \to 0$ as $\mathrm{Im}(v_z) \to -\infty$ (Really $|v_z| \to \infty$)

$$= -2\pi i \sum_j \underset{v_z = v_{zj}}{\mathrm{Res}}[g(v_z)]$$

   d. Thus for pole at $v_z = -ic$

$$= -2\pi i \frac{1}{(-2ic)(-ic - \frac{ip}{k})^2} = \frac{\pi}{c}\frac{-1}{(c+\frac{p}{k})^2}$$

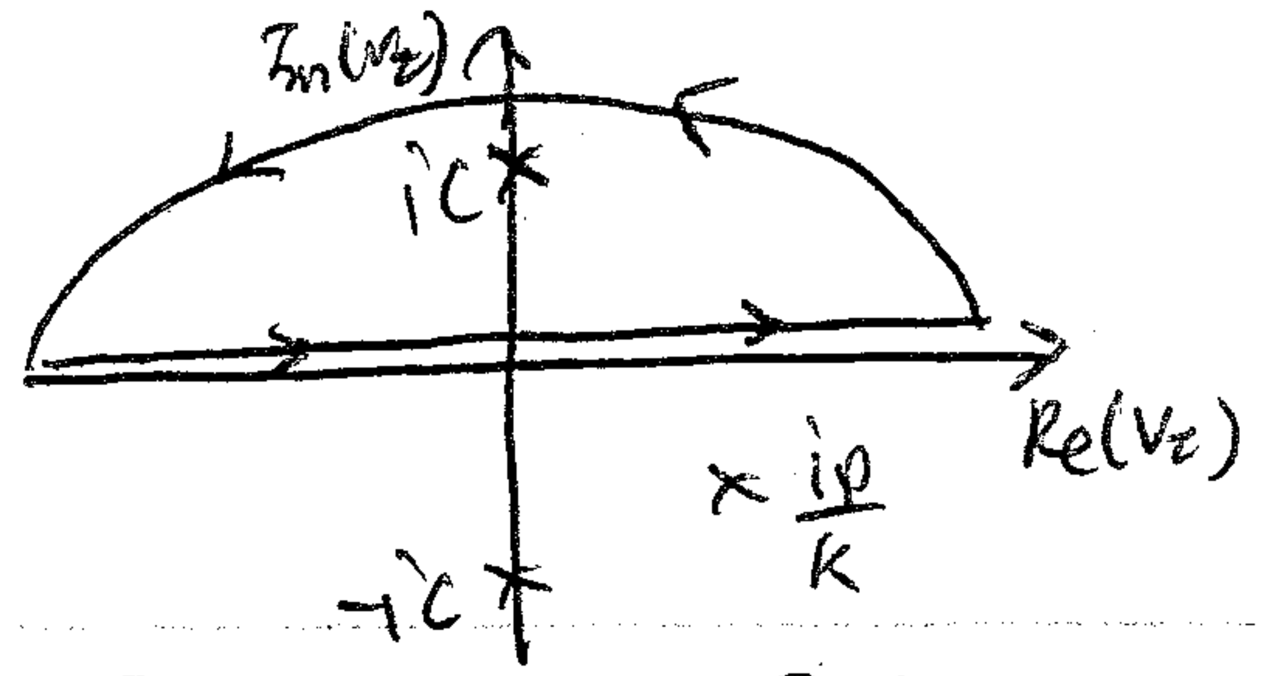
   e. So, we find for $k>0$:

$$D(p,\underline{k}) = 1 + \frac{\omega_p^2}{k^2}\frac{c}{\pi}\frac{\pi}{c}\frac{+1}{(c+\frac{p}{k})^2} = 1 + \frac{\omega_p^2}{(p+kc)^2}$$

Lecture #13 (Continued)

II. B. (Continued).

4. Similarly for $k<0$



a. Close in upper half plane $\mathrm{Im}(v_z) \to \infty$ (CCW orientation).

b. Thus $\int_{-\infty}^{\infty} dv_z\, g(v_z) = 2\pi i \sum_j \underset{v_z = v_{zj}}{\mathrm{Res}} \left[g(v_z) \right]$ $\to$ pole at $v_z = +ic$

$$= 2\pi i \frac{1}{2ic\left(ic - \frac{ip}{k}\right)^2} = \frac{-\pi}{c} \frac{1}{\left(c - \frac{p}{k}\right)^2}$$

c. Thus $D(p, \underset{\sim}{k}) = 1 + \dfrac{\omega_p^2}{(p - kc)^2}$

5. a. Noting that for $k>0$, $k = |k|$ and for $k<0$ $k = -|k|$, we can write these as a single equation $\boxed{D(p, \underset{\sim}{k}) = 1 + \dfrac{\omega_p^2}{(p + |k| c)^2} = 0}$

b. NOTE: Since this solution is a polynomial, analytic continuation to the $\mathrm{Re}(p) < 0$ plane is trivial.

6. Roots of dispersion relation are $\boxed{p = -|k| c \pm i\omega_p}$

C. Solving for $N(\underset{\sim}{k}, p)$

Initial condition on f_{s1}

1. $N(\underset{\sim}{k}, p) = -i \sum_s \dfrac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \dfrac{F_s(\underset{\sim}{k}, \underset{\sim}{v}, 0)}{v_z + ip/k}$

a. If we have a specific form for the initial conditions $F_s(\underset{\sim}{k}, \underset{\sim}{v}, 0)$, then we can perform the integral analogous to the procedure above.

b. An important point is that, as long as $F_s(\underset{\sim}{k}, \underset{\sim}{v}, 0)$ do not have any singularities or discontinuities, the result of the integration will not have any singularities. $\to$ Thus, no poles in $N(\underset{\sim}{k}, p)$

II. C. (Continued)

2. Rather than solve for a specific form of $F_s(\underset{\sim}{k}, \underset{\sim}{v}, 0)$, we note

$$\tilde{\phi}(\underset{\sim}{k}, p) \underbrace{D(\underset{\sim}{k}, p)}_{\text{Dispersion Relation}} = \underbrace{N(\underset{\sim}{k}, p)}_{\text{Initial Conditions}}$$

(see I.E.2.a. earlier)

a. We simply denote $N(\underset{\sim}{k}, p) = \phi(\underset{\sim}{k}, 0)$ since it is determined by the initial conditions.

b. Thus $\tilde{\phi}(k, p) = \dfrac{\phi(\underset{\sim}{k}, 0)}{D(p, \underset{\sim}{k})} = \dfrac{\phi(\underset{\sim}{k}, 0)}{1 + \dfrac{\omega_p^2}{(p+|k|C)^2}} = \dfrac{(p+|k|C)^2 \phi(\underset{\sim}{k}, 0)}{(p+|k|C)^2 + \omega_p^2}$

D. Completing Solution for $\phi(\underset{\sim}{k}, t)$

1. As we solved earlier (I.I.3.), $\phi(\underset{\sim}{k}, t) = \sum_j \underset{p=p_j}{\text{Res}} \left[\tilde{\phi}(\underset{\sim}{k}, p) e^{pt} \right]$

a. Here $\tilde{\phi}(\underset{\sim}{k}, p) e^{pt} = \dfrac{(p+|k|C)^2 \phi(\underset{\sim}{k}, 0) e^{pt}}{(p+|k|C - i\omega_p)(p+|k|C + i\omega_p)}$

Poles at roots $p = -|k|C + i\omega_p$ & $p = -|k|C - i\omega_p$

2. Thus $\phi(k, t) = \dfrac{(\cancel{-|k|C} + i\omega_p + \cancel{|k|C})^2 \phi(\underset{\sim}{k}, 0) e^{-|k|Ct} e^{i\omega_p t}}{(\cancel{-|k|C} + i\omega_p + \cancel{|k|C} + i\omega_p)}$

$$+ \frac{(\cancel{-|k|C} - i\omega_p + \cancel{|k|C})^2 \phi(\underset{\sim}{k}, 0) e^{-|k|Ct} e^{-i\omega_p t}}{(\cancel{-|k|C} - i\omega_p + \cancel{|k|C} - i\omega_p)}$$

$$= \frac{-\cancel{\omega_p^2} \phi(\underset{\sim}{k}, 0) e^{-|k|Ct} e^{i\omega_p t}}{2i\cancel{\omega_p}} + \frac{-\cancel{\omega_p^2} \phi(\underset{\sim}{k}, 0) e^{-|k|Ct} e^{-i\omega_p t}}{-2i\cancel{\omega_p}}$$

$$\boxed{\phi(\underset{\sim}{k}, t) = -\phi(\underset{\sim}{k}, 0) e^{-|k|Ct} \left(\frac{e^{i\omega_p t} - e^{-i\omega_p t}}{2i} \right)}$$