I. Kinetic Stability of a Plasma

A. **Gardner's Theorem**: A single-humped velocity distribution is always stable.

Proof: 1. Roots of dispersion relation $D(\underline{k}, p) = 0$ give real frequency and growth/decay rates of normal modes.

2. For a solution with $Re(p) > 0$, the plasma is unstable!

3. Proof by contradiction: Assume there are solutions with $Re(p) > 0$.

4. Since $Re(p) > 0$, then, for $k > 0$, we can take to contour of integration along the $Re(v_z)$ axis

$$D(\underline{k}, p) = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{v_z - \frac{\omega}{k} - \frac{i\gamma}{k}} = 0$$

where we have substituted $p = \gamma - i\omega$

5a. We can separate the integrand into Real & Imaginary parts

$$\frac{\partial F_0/\partial v_z}{v_z - \frac{\omega}{k} - i\frac{\gamma}{k}} \frac{(v_z - \frac{\omega}{k} + i\frac{\gamma}{k})}{(v_z - \frac{\omega}{k} + i\frac{\gamma}{k})} = \frac{\frac{\partial F_0}{\partial v_z}(v_z - \frac{\omega}{k})}{(v_z - \frac{\omega}{k})^2 + \frac{\gamma^2}{k^2}} + i\frac{\gamma}{k} \frac{\partial F_0/\partial v_z}{(v_z - \frac{\omega}{k})^2 + \frac{\gamma^2}{k^2}}$$

b. Thus

$$D_r(\underline{k}, p) = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z (v_z - \frac{\omega}{k})}{(v_z - \frac{\omega}{k})^2 + (\frac{\gamma}{k})^2} = 0$$

$$D_i(\underline{k}, p) = -\frac{\omega_p^2}{k^2}\frac{\gamma}{k} \int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{(v_z - \frac{\omega}{k})^2 + (\frac{\gamma}{k})^2} = 0$$

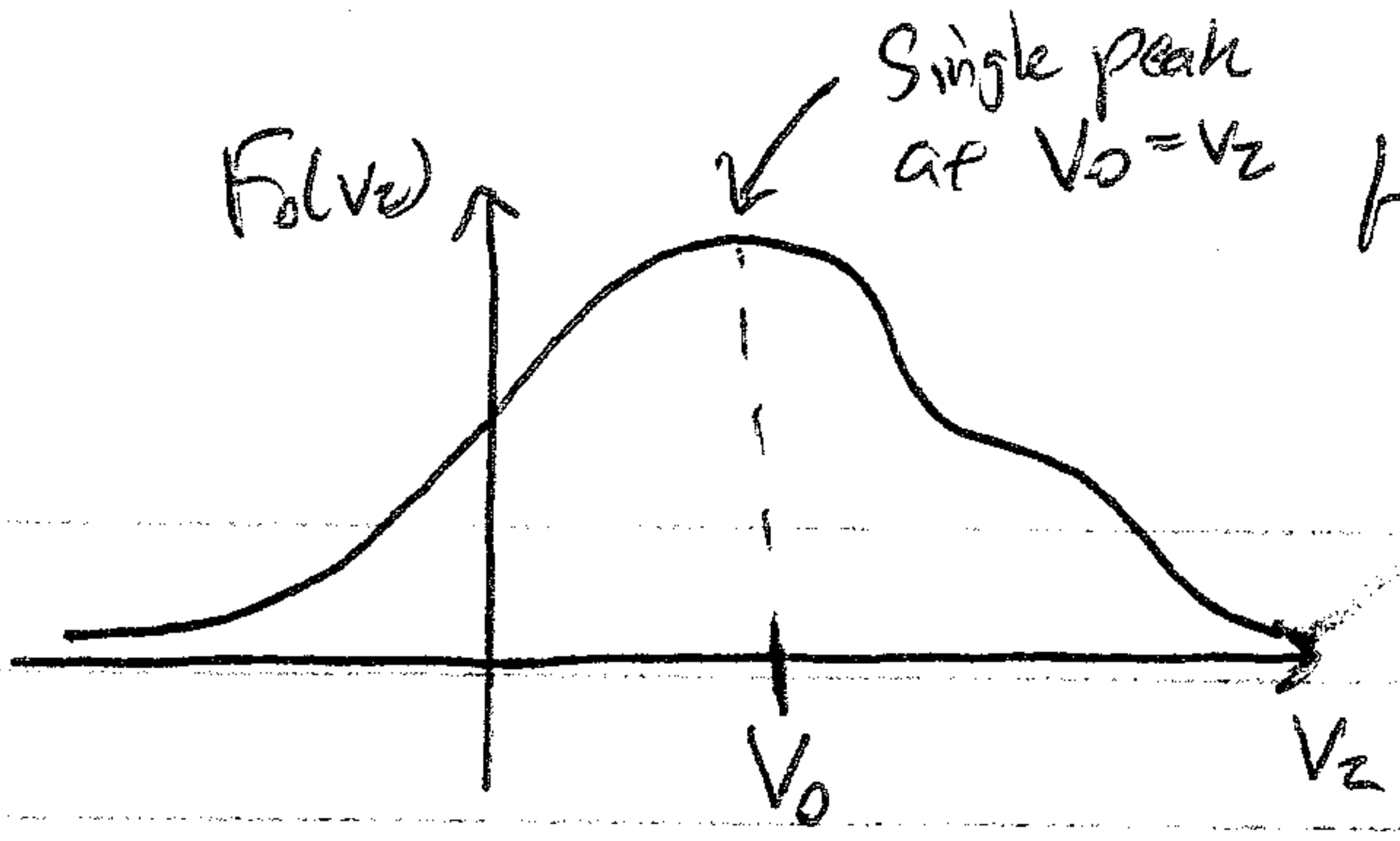
c. Both real & imaginary pieces must equal zero separately. Thus any linear combination of D_r & D_i must equal zero.

Take $D_r - \left(\frac{kV_0 - \omega}{\gamma}\right) D_i = 0$

Lecture #16 (Continued) Howes ③

I. A. (Continued)

6. Single-humped velocity distribution



7. Thus $D_r + \left(\frac{kV_0-\omega}{\gamma}\right) D_i = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{(v_z-\frac{\omega}{k})^2+(\frac{\gamma}{k})^2}\left[\left(v_z-\frac{\omega}{k}\right)-\left(\frac{kV_0-\omega}{\gamma}\right)\frac{\gamma}{k}\right]$

a. Pieces in brackets $[\;\;] = v_z - \cancel{\frac{\omega}{k}} - V_0 + \cancel{\frac{\omega}{k}} = v_z - V_0$

b. Thus, we have

$$1 + \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{\left(\frac{\partial F_0}{\partial v_z}\right)(V_0 - v_z)}{(v_z-\frac{\omega}{k})^2+(\frac{\gamma}{k})^2} = 0$$

8. NOTE: a. Denominator of integrand is positive definite

b. For single humped distribution

$$\left(\frac{\partial F_0}{\partial v_z}\right)(V_0 - v_z) > 0 \quad \text{for } v_z > V_0 \text{ and } v_z < V_0.$$

9. a. Thus, the integrand is positive definite, leading to a positive definite integral. Thus, the relation above can never be satisfied!

b. This contradiction means the original assumption, $Re(p) > 0$, is false.

c. <u>Thus, the single-humped distribution is always stable.</u>

QED.

B. <u>Nyquist Criterion</u>

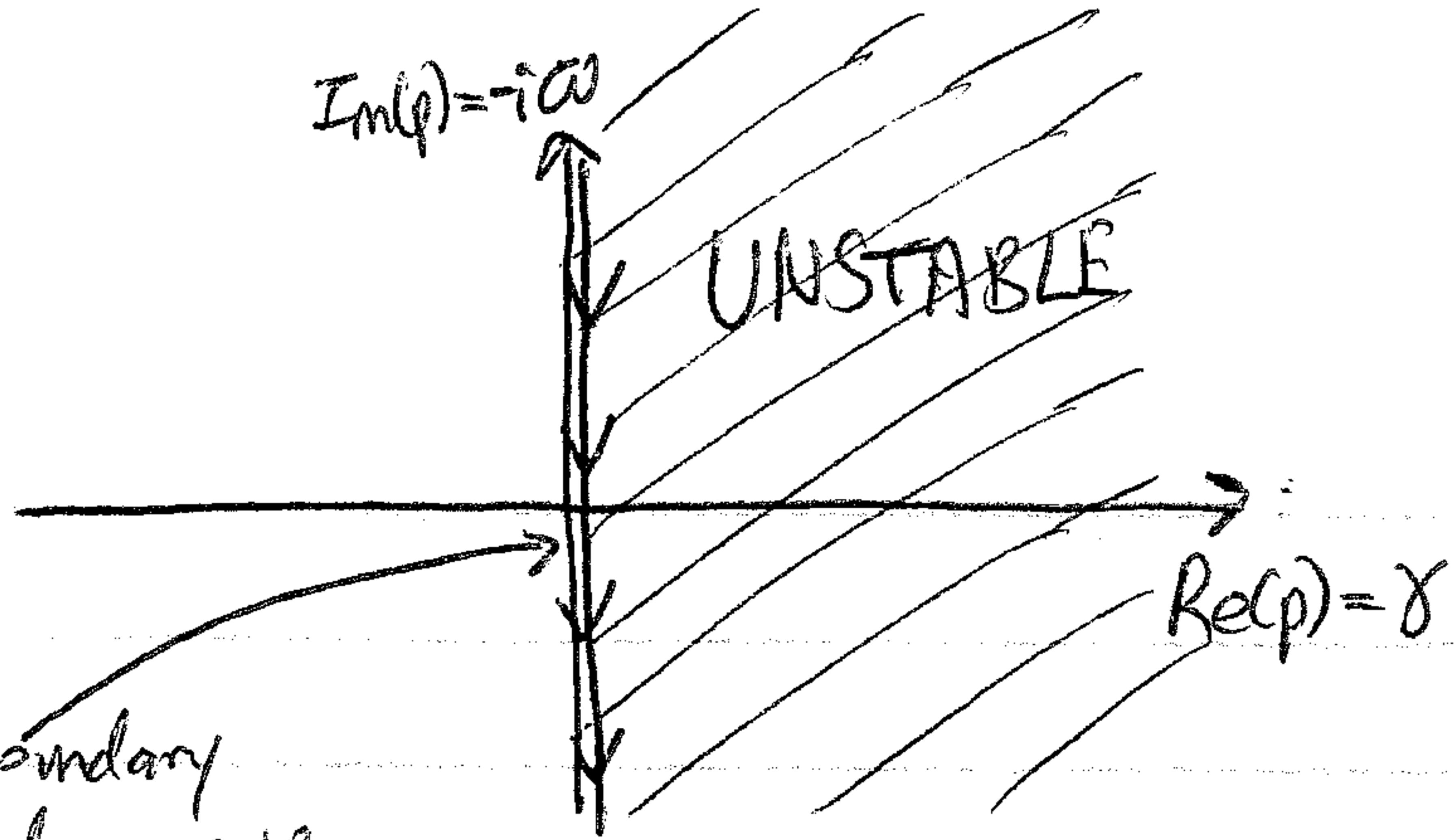
1. If a distribution function with a single peak is stable, how do we test a multiply-humped distribution for stability?

2. Dispersion Relation: $D(\underline{k}, p) = 0$ yields solutions.
If some $\underline{k}$ exists such that a solution has $Re(p) = \gamma > 0$, UNSTABLE!

Lecture #16 (Continued)

I. B. (Continued)

3. Complex p-plane



a. Line $\gamma = 0$ is boundary between stable and unstable.
$\Rightarrow$ Marginal Stability

b. Since $D(\underline{k}, p)$ is a complex function of p, we can map the unstable $(Re(p) > 0)$ half of the p-plane into complex-D space.

c. This unstable half-plane is bounded by the $\gamma = 0$ line from $\omega = -\infty$ to $\omega = +\infty$.

d. If the point $D = 0$ falls within the mapping of unstable region, the an unstable solution exists.

4. Example: a. Cauchy Velocity Distribution (II.B.5) $F_0(v_z) = \frac{C}{\pi} \frac{1}{(C^2 + v_z^2)}$

b. From lecture #13, the dispersion relation is

$$D(\underline{k}, p) = 1 + \frac{\omega_p^2}{(p + |k|C)^2}$$

c. Substituting $p = \gamma - i\omega$ and calculating D_r and D_i gives

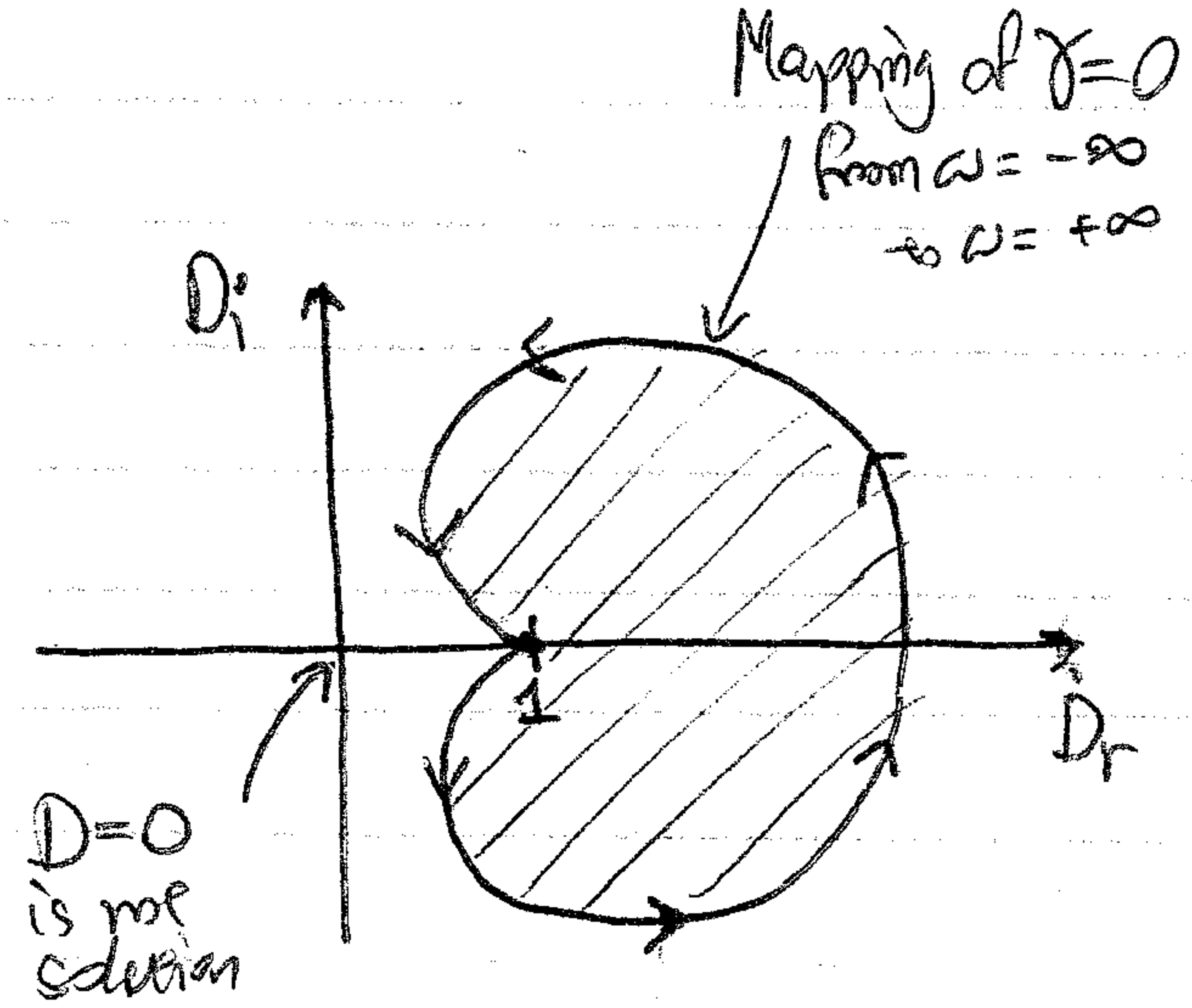
$$D_r = 1 + \frac{\omega_p^2 \left[(\gamma + |k|C)^2 - \omega^2 \right]}{\left[(\gamma + |k|C)^2 + \omega^2 \right]^2}$$

$$D_i = \frac{\omega_p^2 \, 2|k|C\omega}{\left[(\gamma + |k|C)^2 + \omega^2 \right]^2}$$

d. Setting $\gamma = 0$ (Marginal Stability) we obtain:

$$D_r = 1 + \frac{\omega_p^2 (k^2C^2 - \omega^2)}{\left[k^2C^2 + \omega^2 \right]^2}$$

$$D_i = 2 \frac{\omega_p^2 |k| C \omega}{\left[k^2C^2 + \omega^2 \right]^2}$$



I. B.4. (Continued)

e. Since $D=0$ is not within the mapping of the unstable $Re(p)>0$ half-plane, the plasma is stable.

$\Rightarrow$ This result is consistent with Gardner's Theorem.

C. <u>The Winding Theorem</u>: If a closed contour C_p in the complex p-plane encloses n simple zeros of some mapping function $D(p)$, then the corresponding contour C_D in the complex D-plane must make n turns around the origin.

<u>Proof</u>:

1. From Residue theorem, number of turns N_W of contour C_D about the origin is

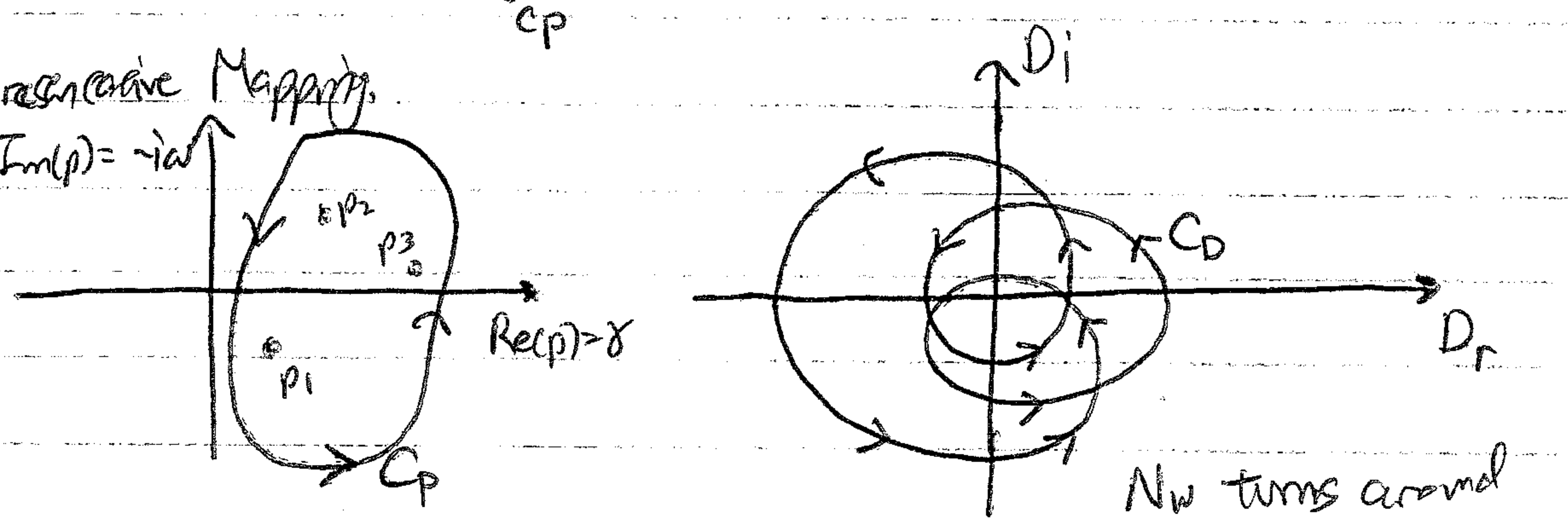
$$N_W = \frac{1}{2\pi i}\int_{C_D} \frac{dD}{D}$$ pole occurs at $D=0$.

<u>Def: Winding Number</u>

2. Changing variables to the p-plane $\left(dD = \frac{\partial D}{\partial p} dp \right)$, we have

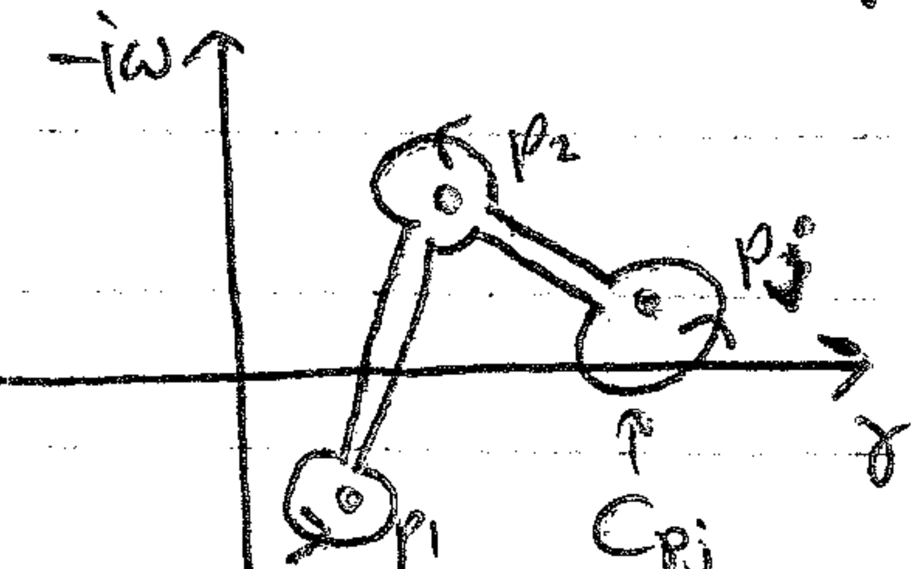
$$N_W = \frac{1}{2\pi i}\int_{C_p} \frac{1}{D}\frac{\partial D}{\partial p}\, dp$$

3. Representative Mapping:



N_W turns around $D = D_r + iD_i = 0$

4. Deform Contour C_p:



$$N_W = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_{C_{p_j}} \frac{1}{D}\frac{\partial D}{\partial p}\, dp$$

I. C. (Continued)

5. a. Taylor Expand $D(p)$ about $p=p_j$

$$D(p) = \cancelto{0}{D(p_j)} + \left.\frac{\partial D}{\partial p}\right|_{p_j}(p-p_j) + \dots$$

Solution

b. We can also expand the function $g=\frac{\partial D}{\partial p}$ about $p=p_j$

$$g(p) = g(p_j) + (p-p_j)\left.\frac{\partial g}{\partial p}\right|_{p_j} + \dots$$

To lowest order, keep only $g(p_j) \Rightarrow \quad \frac{\partial D}{\partial p} = \left.\frac{\partial D}{\partial p}\right|_{p_j}$

c. Thus $\quad \frac{1}{D}\frac{\partial D}{\partial p} = \frac{\cancel{\left(\left.\frac{\partial D}{\partial p}\right|_{p_j}\right)}}{(p-p_j)\cancel{\left(\left.\frac{\partial D}{\partial p}\right|_{p_j}\right)}} = \frac{1}{p-p_j}$

6. Thus, we obtain $\quad N_W = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_{C_{p_j}} \frac{dp}{p-p_j} = n \quad \checkmark \qquad$ QED.

D. The Penrose Condition

1. Let's Apply the Nyquist Criterion to a distribution function with an arbitrary number of humps.

2. a. We want an expression for $D(\underline{k},p)$ valid for any distribution function.

b. Since γ is always small near $\gamma=0$, we can use the Plemelj Relation to evaluate $D_r(\underline{k},p)$ and $D_i(\underline{k},p)$. (We did this for Weak Growth Rate Approximation, Lect #14, II.C.4.)

$$D_r(\underline{k},\omega) = 1 - \frac{\omega_p^2}{k^2} P\int_{-\infty}^{\infty} dv_z \frac{\partial f_0/\partial v_z}{v_z - \frac{\omega}{k}}$$

$$D_i(\underline{k},\omega) = -\pi \frac{k}{|k|}\frac{\omega_p^2}{k^2}\left.\frac{\partial f_0}{\partial v_z}\right|_{v_z=\frac{\omega}{k}}$$

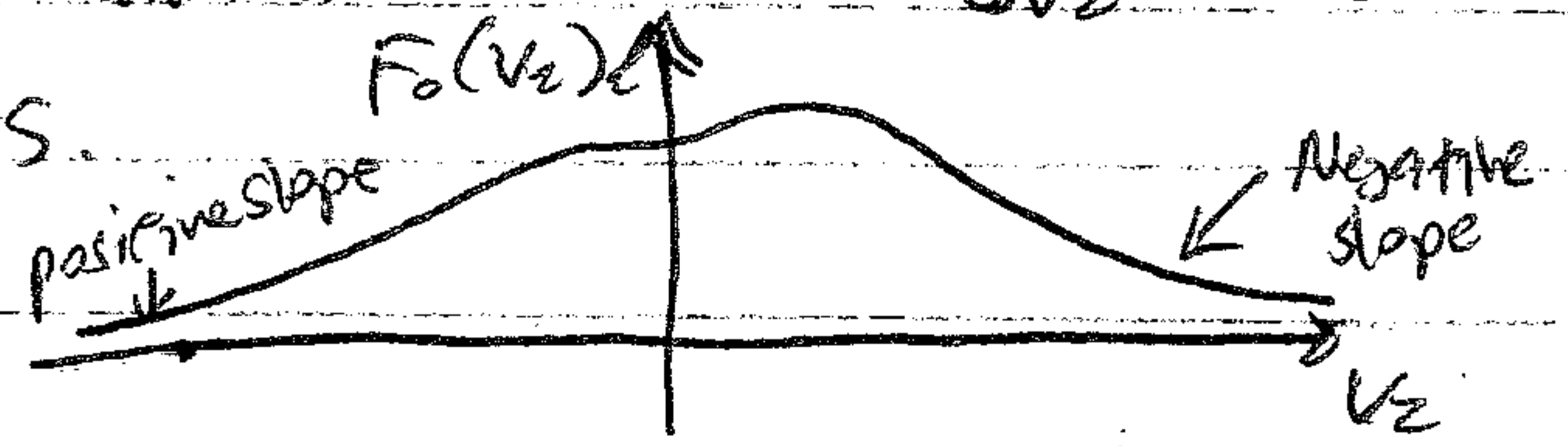
I. D. (Continued)

3. Shape of $\gamma = 0$ curve in D-plane near $\omega = \pm\infty$

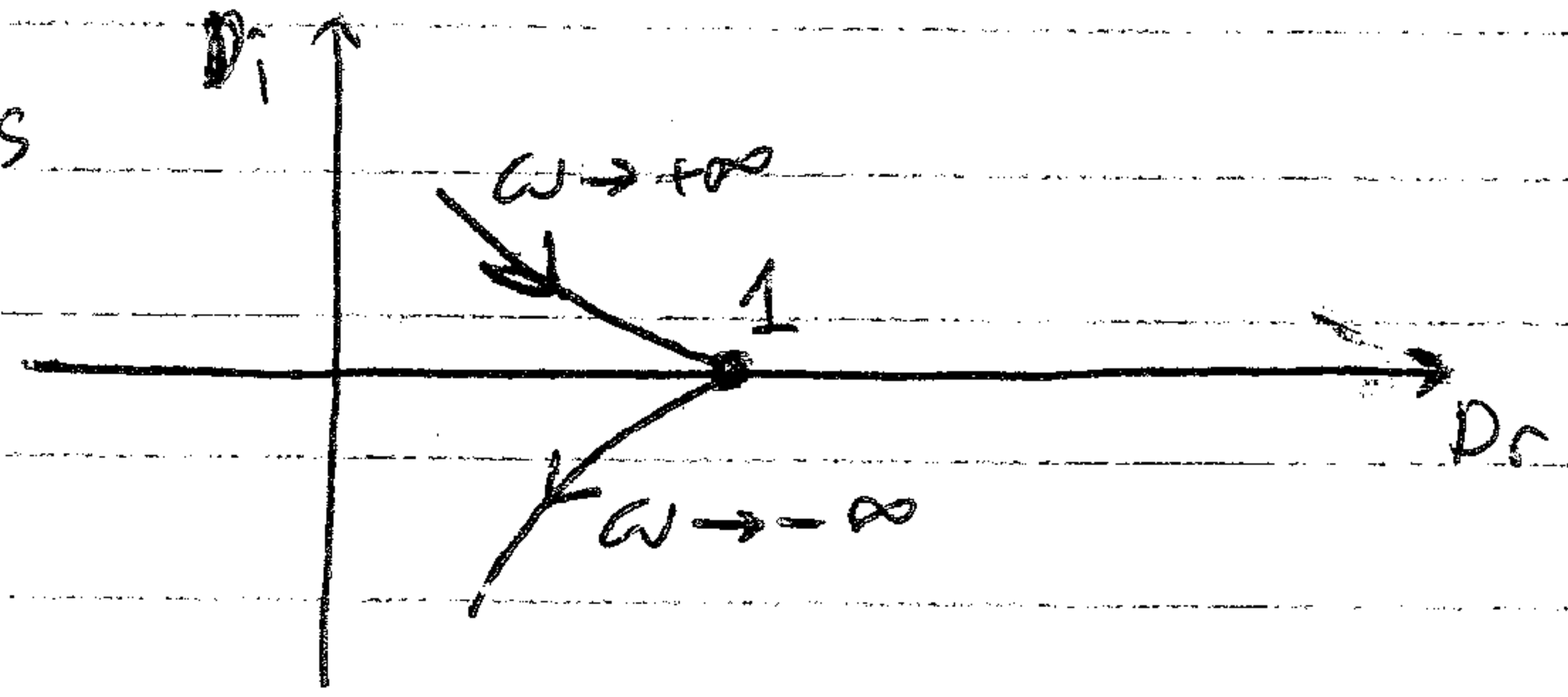
a. First, we'll assume $k > 0$.

b. NOTE: $\frac{\partial F_0}{\partial v_z} \to 0$ as $\omega \to \infty$, so $\lim_{\omega \to \infty} D_r = 1$ $\lim_{\omega \to \infty} D_i = 0$.

c. Also, as $\omega \to +\infty$ $\frac{\partial F_0}{\partial v_z} < 0$ and as $\omega \to -\infty$ $\frac{\partial F_0}{\partial v_z} > 0$.
since $F_0(v_z) > 0$ always.



d. Thus

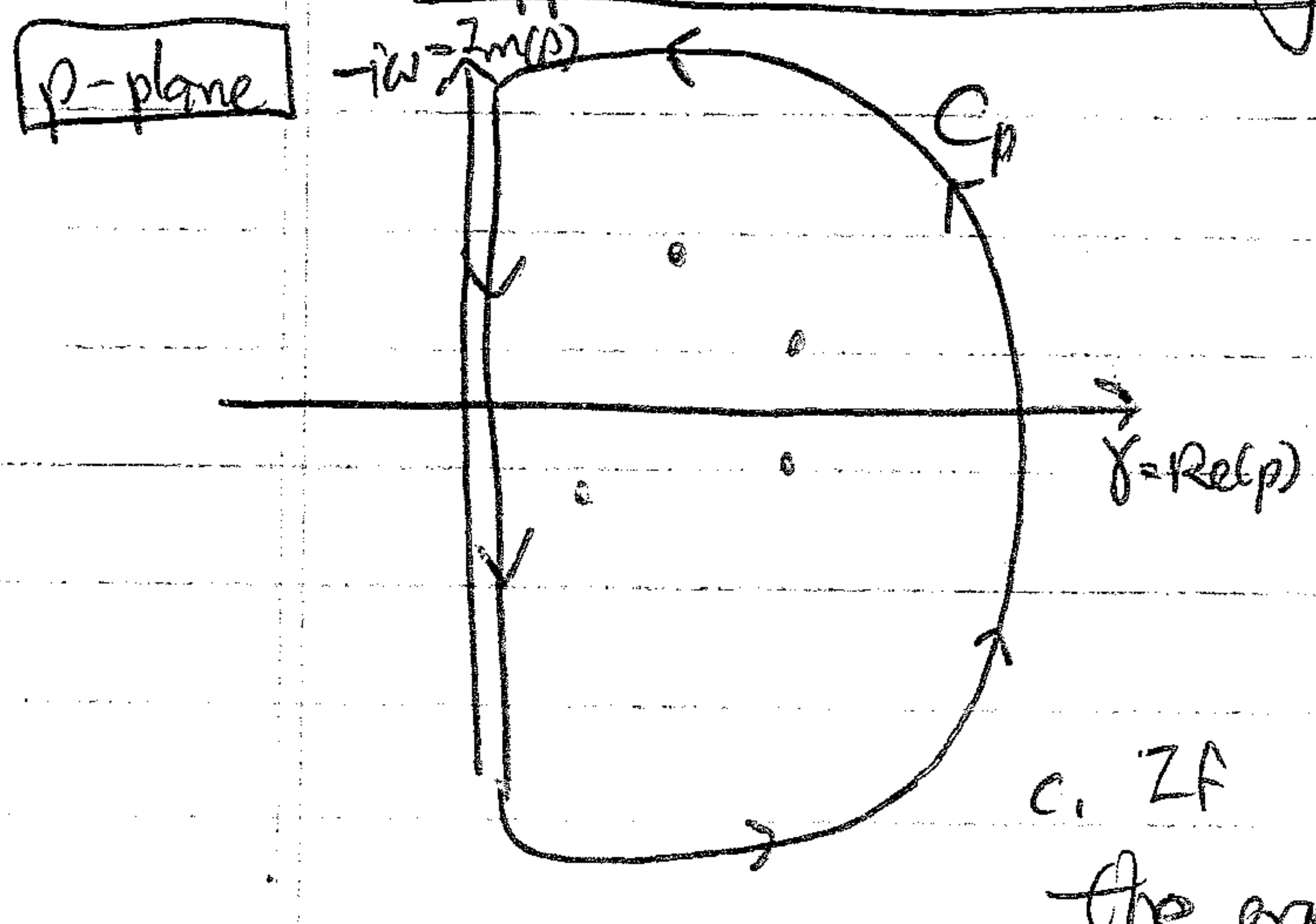


4. Crossing D_r-axis ($D_i = 0$)

a. Contour crosses D_r axis when $D_i = 0 \Rightarrow \frac{\partial F_0}{\partial v_z} = 0$
$\Rightarrow$ Only crosses D_r where distribution function has zero slope.

b. For a smooth, continuous $F_0(v_z)$, there are always an odd number of crossings.

5. Application of the Winding Theorem



a. Take contour C_p downward along $\gamma = 0$, closing at infinity in right half plane.
(Thus C_p encloses all unstable roots)

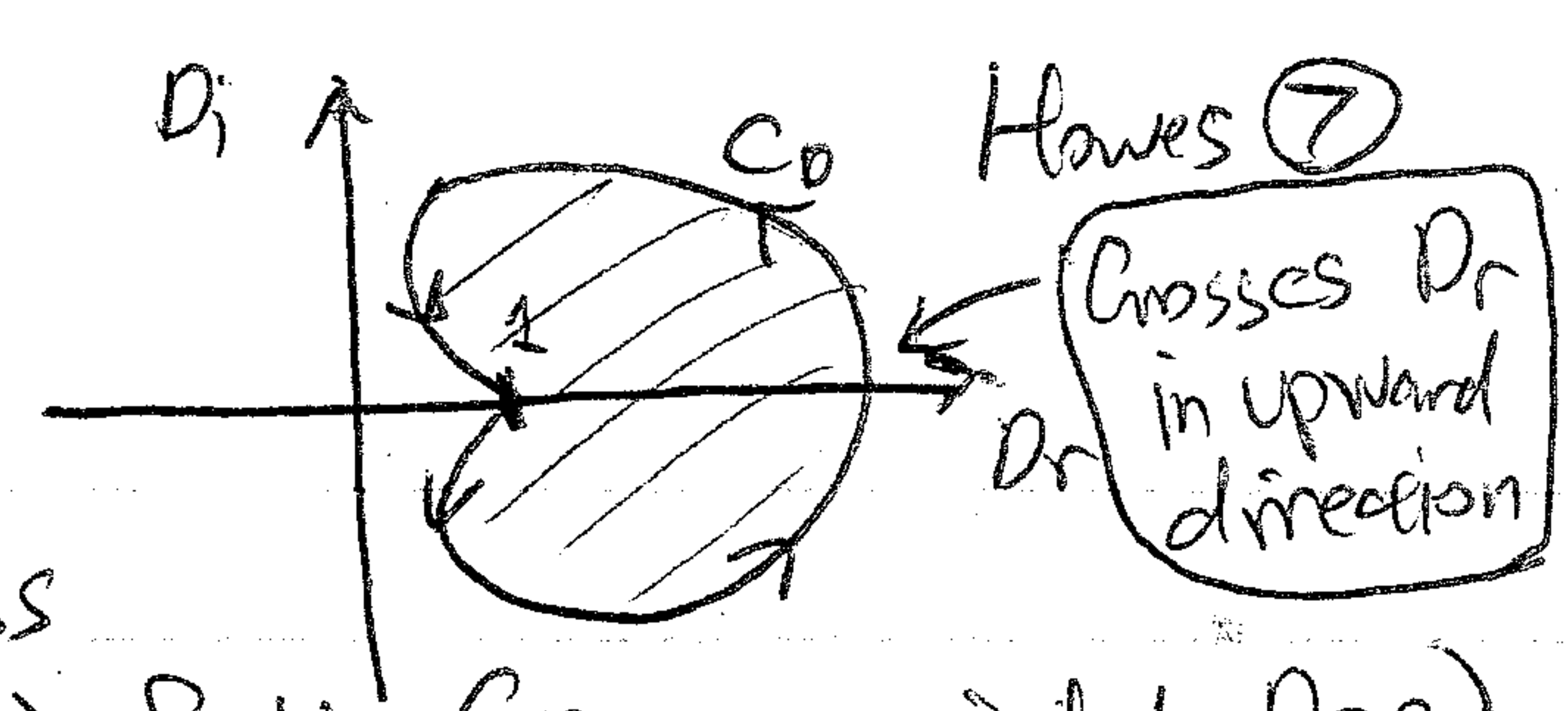
b. Contour at $|p| = \infty$ maps to $D = 1$.
Rest of contour maps to $\gamma = 0$ curve.

c. If $\gamma = 0$ curve in D-space winds around the origin ($D = 0$), (CCW), then unstable root exists.

Lecture #16 (Continued) Howes ⑦

Z. D. (Continued)

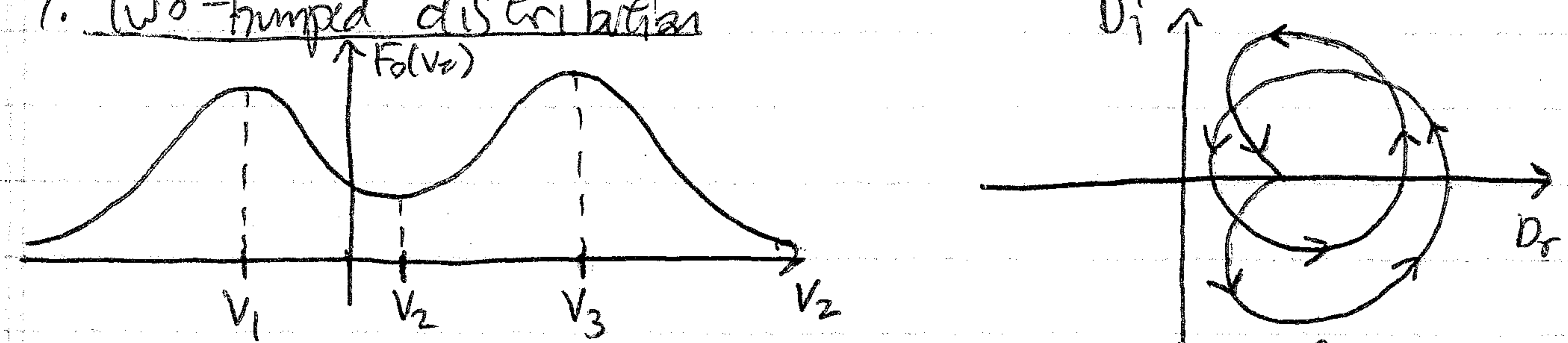
6. For single-hump distribution:



a. CCW contour C_D must cross D_r axis to the right of 1 ⇒ Stable (Does not include $D=0$).

⇒ Once again, we have proven Gardner's Theorem.

7. Two-humped distribution



a. Cross D_r-axis three times: Upward at V_1 & V_3 (maxima)
Downward at V_2 (minimum)

b. At the points where $\frac{\partial F_0}{\partial v_z} = 0$, $D_i = 0$ and $v_z = \frac{\omega}{k} = V_j$.

Thus, the crossing of the D_r-axis occurs at

$$D_r = 1 - \frac{\omega_p^2}{k^2} P\int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{v_z - V_j}$$

c. If $D_r < 0$ at a crossing, the plasma is unstable by the Nyquist criterion.

d. Since k may be arbitrarily small, we must have

$$P\int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{v_z - V_j} > 0 \quad \text{for any } j \text{ to have instability.}$$

8. The Penrose Condition

a. Noting that $F_0(V_j) = $ const, we may write

$$P\int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{v_z - V_j} = P\int_{-\infty}^{\infty} dv_z \frac{\frac{\partial}{\partial v_z}[F_0(v_z) - F_0(V_j)]}{v_z - V_j}$$

I. D. 8. (Continued)

b. Integrating by parts, we obtain

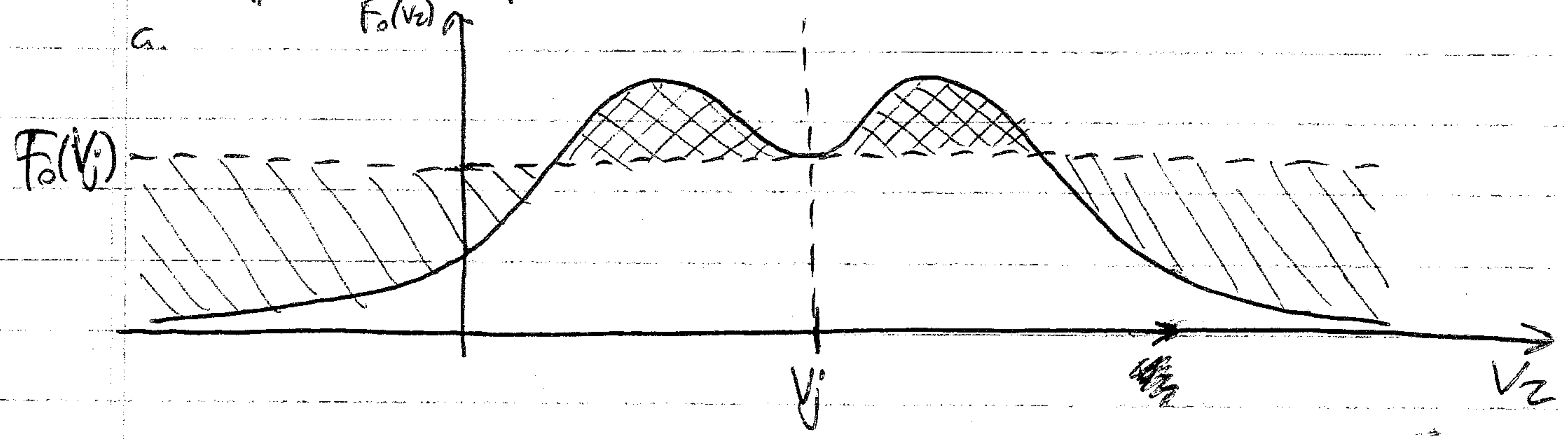
$$\boxed{\int_{-\infty}^{\infty} dv_z \frac{F_0(v_z) - F_0(V_j)}{(v_z - V_j)^2} > 0}$$ Penrose Condition for Instability

c. NOTE! We may drop principal value since numerator $= 0$ when $v_z = V_j$.

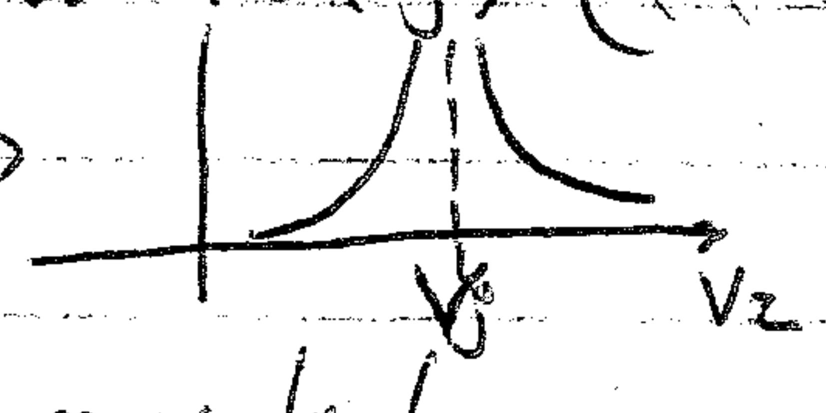
d. Penrose Condition applies for a distribution function with any number of humps.

9. Graphical Interpretation of Penrose Condition

a.



b. Integral is ~~weighting~~ summation of distribution above $F_0(V_j)$ (cross-hatched) minus that below $F_0(V_j)$ (hatched) weighted by function $\frac{1}{(v_z - V_j)^2}$ $\Rightarrow$



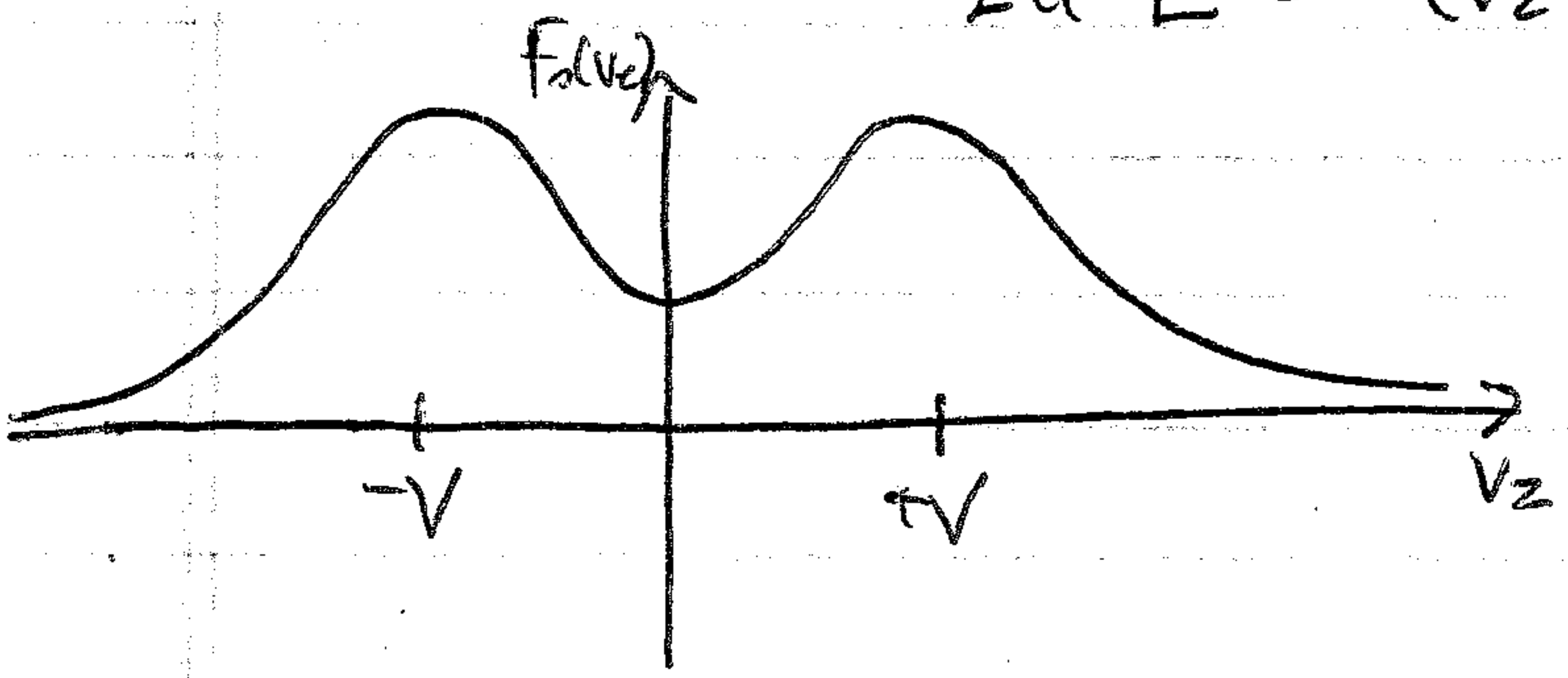
c. Thus, humps above minimum at $F_0(V_j)$ must be large enough that integral is positive.

I. (Continued) **Examples**

E. Counter-Streaming Beam Instability

1. **DEF:** Counter-Streaming Cauchy Distribution

$$F_0(v_z) = \frac{c}{2\pi}\left[\frac{1}{c^2+(v_z-V)^2} + \frac{1}{c^2+(v_z+V)^2}\right]$$



a. For $\lim_{c\to 0} F_0(v_z)$ we get two counter-streaming delta function beams. The "zero" temperature limit. UNSTABLE

b. As c increases, eventually the distribution transitions to a a single hump at $c=\sqrt{3}V$ STABLE by Gardner's Thm.

c. Thus, at some point between $c=0$ and $c=\sqrt{3}V$ the system goes from unstable to stable.
(For increasing temperature, the distribution becomes stable).

2. Apply Penrose Condition

a. At peaks at $v_z=\pm V$, Penrose Condition is clearly negative.

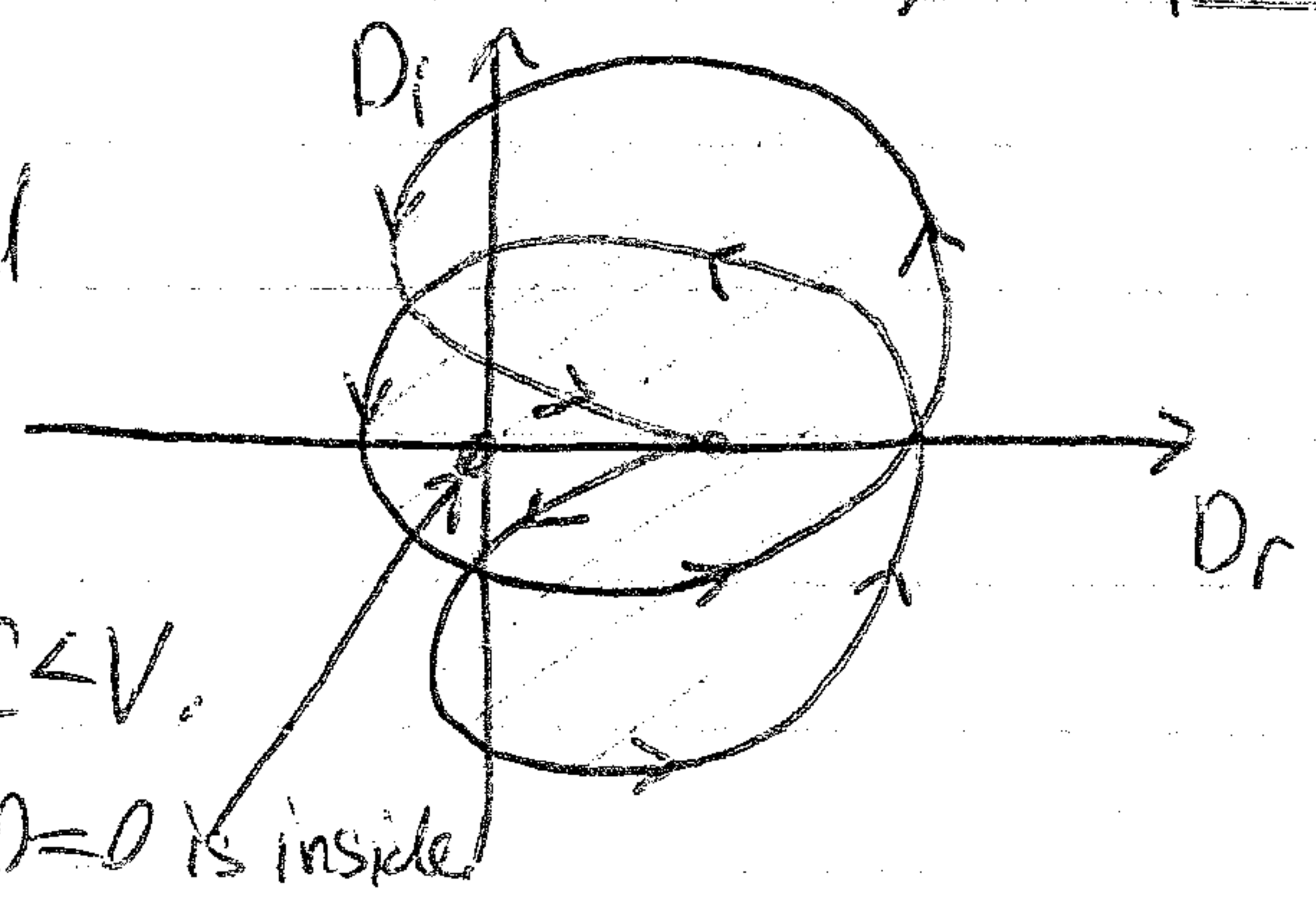
b. At $v_z=0$, we can show that

$$\int_{-\infty}^{\infty} dv_z \frac{F_0(v_z)-F_0(0)}{(v_z-0)^2} = \frac{V^2-c^2}{(V^2+c^2)^2}$$

c. Thus Plasma is unstable when $V^2-c^2>0$, or Unstable $\boxed{V>c}$

3. Nyquist Criterion

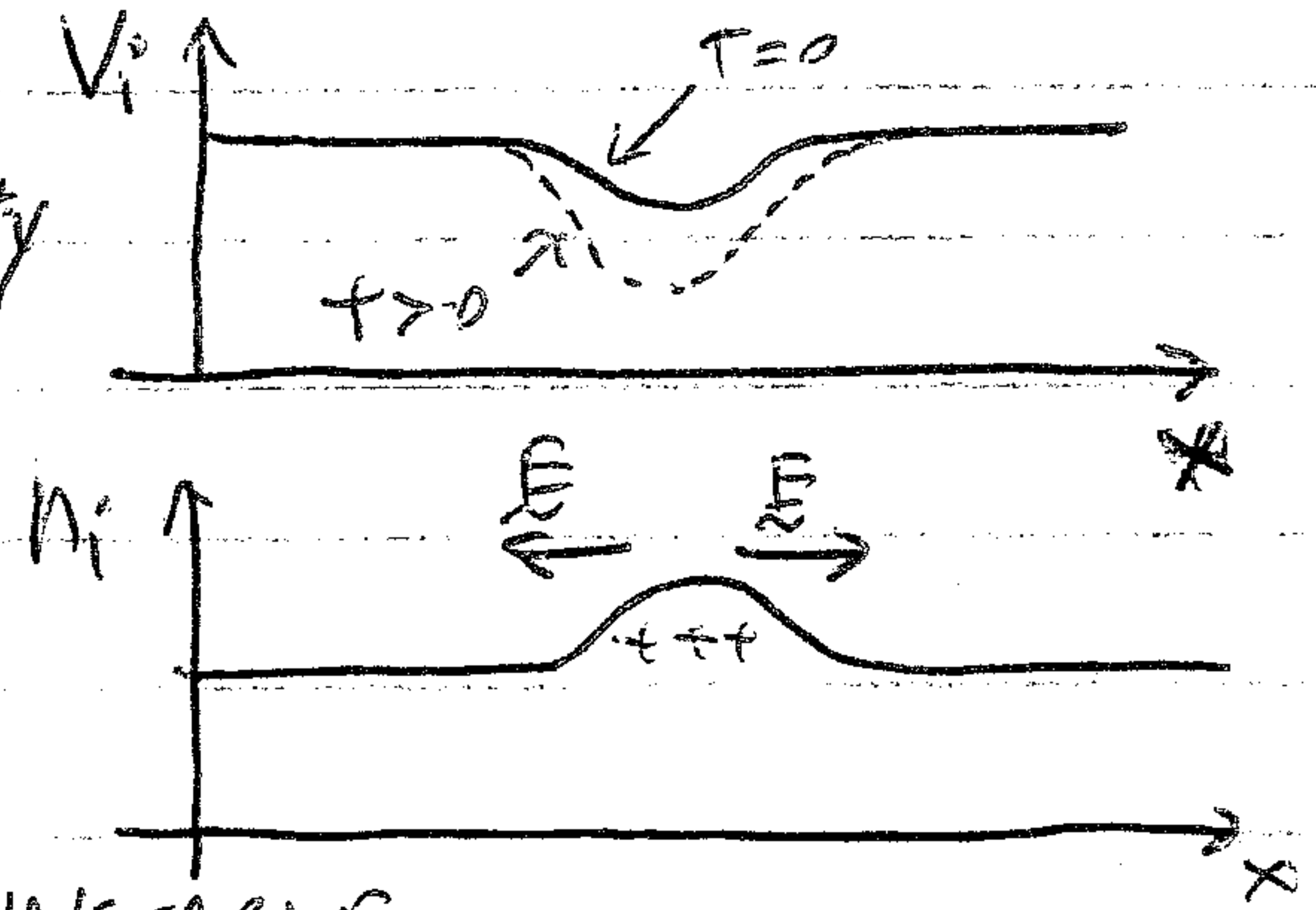
a. We can also evaluate D_r and D_i for this distribution to show the $\gamma=0$ curve gives an unstable root for $c<V$.



II. Overview of Fluid vs. Kinetic Instabilities

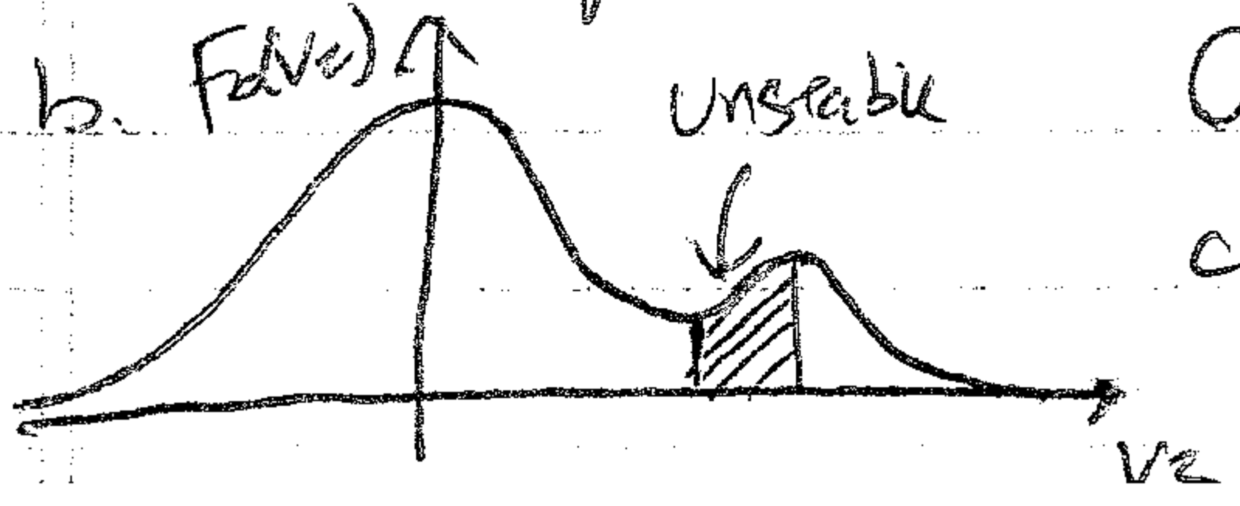
A. Fluid vs. Kinetic Instabilities:

1. The two-stream instability of cold beams is a fluid instability, because all particles react in the same way (no thermal spread of velocities).
2. Such fluid instabilities can be studied by fluid equations.
3. Physical Picture of Two-Stream Instability
   a. Consider a beam of positive charges streaming through a neutralizing background at rest.
   b. Conservation of number density means $n_i V_i$ = constant.
   c. If a ~~pert~~ perturbation leads to a decrease in beam velocity V_i at some point, the density of ions must increase.
   d. The resulting electric field leads to a further slowing of the beam ⇒ Positive feedback ⇒ UNSTABLE



   e. Such an instability will continue to grow until some previously neglected nonlinear term halts the growth.

4. Kinetic Instability
   a. ~~A~~ The free energy in a finite temperature distribution function can lead to growth of instability due to interaction with resonant particles.
   b. Only resonant particles are affected!



   c. Eventually free energy is tapped and kinetic instability saturates →