Lecture #16  Kinetic Stability of a Plasma

I. Kinetic Stability of a Plasma

A. Gardner's Theorem: A single-humped velocity distribution is always stable.

Proof: 1. Roots of dispersion relation $D(k, p) = 0$ give real frequency and growth/decay rates of normal modes.

2. For a solution with $\text{Re}(p) > 0$, the plasma is unstable.

3. Proof by contradiction: Assume there are solutions with $\text{Re}(p) > 0$.

4. Since $\text{Re}(p) > 0$, then, for $k > 0$, we can take to linear of integration along the $\text{Re}(v_k)$ axis

$$D(k, p) = 1 - \frac{c v_k^2}{k^2} \int_{-\infty}^{\infty} dv_k \frac{\frac{\partial F}{\partial v_k}}{v_k - \frac{\omega}{k} - i \frac{\gamma}{k}} = 0$$

where we have substituted $p = \gamma - i \omega$

5a. We can separate the integrand into Real & Imaginary parts

$$\frac{\partial F}{\partial v_k} = \frac{\partial F}{\partial v_k} \left( \frac{\omega}{k} - \frac{\gamma}{k} \right) + i \frac{\partial F}{\partial v_k} \left( \frac{\omega}{k} - \frac{\gamma}{k} \right)$$

6. Thus

$$D_r (k, p) = 1 - \frac{c v_k^2}{k^2} \int_{-\infty}^{\infty} dv_k \frac{\frac{\partial F}{\partial v_k} \left( \frac{\omega}{k} - \frac{\gamma}{k} \right)}{(v_k - \frac{\omega}{k})^2 + \left( \frac{\gamma}{k} \right)^2} = 0$$

$$D_i (k, p) = -\frac{c v_k^2}{k^2} \int_{-\infty}^{\infty} dv_k \frac{\frac{\partial F}{\partial v_k} \left( \frac{\omega}{k} - \frac{\gamma}{k} \right)}{(v_k - \frac{\omega}{k})^2 + \left( \frac{\gamma}{k} \right)^2} = 0$$

7. Both real & imaginary pieces must equal zero separately. Thus any linear combination of $D_r$ & $D_i$ must equal zero.

Take $D_r - \left( \frac{k v_k - c v}{\gamma} \right) D_i = 0$
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I. A. (Continued)

6. Single-humped velocity distribution

7. Thus, \( \Delta_c + \left( \frac{kV_0 - \omega}{\alpha} \right) \Delta_i = 1 - \frac{\alpha^2}{k^2} \int_{-\infty}^{\infty} dv_2 \frac{\delta F_0}{\delta v_2} \left[ (v_2 - \frac{\omega}{k})^2 - \left( \frac{kV_0 - \omega}{k} \right)^2 \right] \)

a. Pieces in brackets \( \left[ - \right] = v_2 - \frac{\omega}{k} - V_0 + \frac{\omega}{k} = V_2 - V_0 \)

b. Thus, we have
\[
1 + \frac{\alpha^2}{k^2} \int_{-\infty}^{\infty} dv_2 \frac{\delta F_0}{\delta v_2} \frac{(V_0 - V_2)}{(V_2 - \frac{\omega}{k})^2 + (\frac{\omega}{k})^2} = 0
\]

8. **Note**:
   a. Denominator of integrand is positive definite
   b. For single humped distribution
\[
\left( \frac{\delta F_0}{\delta v_2} \right) (V_0 - V_2) > 0 \quad \text{for } V_2 > V_0 \text{ and } V_2 < V_0.
\]

9. a. Thus, the integrand is positive definite, leading to a positive definite integral. Thus, the relation above can never be satisfied!

b. This contradiction means the original assumption, \( \text{Re}(p) > 0 \), is false.

c. Thus, the single-humped distribution is **always stable**.

Q.E.D.

B. Nyquist Criterion

If a distribution function will a single peak is stable, how do we test a multiply-humped distribution for stability?

2. Dispersion Relation: \( D(\Delta, p) = 0 \) yields solutions.

If some \( \Delta \) causes such that a solution has \( \text{Re}(p) = \Re > 0 \), UNSTABLE!
Lecture 16 (Continued)

2. Complex p-plane

\[ \text{Im}(p) = \pm \omega \]

\[ \text{Re}(p) = \gamma \]

\[ \text{UNSTABLE} \]

a. Line \( \gamma = 0 \) is boundary between stable and unstable, 
   \( \implies \text{Marginal Stability} \)

b. Since \( D(k, p) \) is a complex function of \( p \), we can map the unstable \( (\text{Re}(p) > 0) \) half of the p-plane into complex-D space.

c. This unstable half-plane is bounded by the \( \gamma = 0 \) line 
   from \( \omega = -\infty \) to \( \omega = +\infty \).

d. If the point \( D = 0 \) falls within the mapping of unstable region, the an unstable solution exists.

   \[ F_0(v) = \frac{C}{\pi} \frac{1}{(C^2 + v^2)} \]

b. From lecture #13, the dispersion relation is
   \[ D(k, \omega) = 1 + \frac{a_k \omega^2}{(\omega^2 + k^2)^2} \]

c. Substituting \( \omega = \gamma - \omega \) and calculating \( D_r \) and \( D_i \) gives
   \[ D_r = 1 + \frac{a_k \omega^2 (\omega + k \omega - \omega^2)}{[\omega^2 + k^2 \omega^2]^2} \]

   \[ D_i = \frac{a_k \omega^2 2 k \omega}{[\omega^2 + k^2 \omega^2]^2} \]

d. Sketching \( \gamma = 0 \) (Marginal Stability) 
   we obtain:
   \[ D_r = 1 + \frac{a_k \omega^2 (k^2 + \omega^2)}{[k^2 + \omega^2]^2} \]
   \[ D_i = 2 \frac{a_k \omega^2 k \omega}{[k^2 + \omega^2]^2} \]

   \[ D = 0 \text{ is not a solution} \]

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C. The Winding Theorem: If a closed contour $C_p$ in the complex $p$-plane encloses $n$ simple zeros of some mapping function $D(p)$, then the corresponding contour $C_D$ in the complex $D$-plane must make $n$ turns around the origin.

Proof:

1. From Residue theorem, number of turns of contour $C_D$ above the origin is

$$N_w = \frac{1}{2\pi i} \oint_{C_D} \frac{dD}{D}$$

Def: Winding Number

2. Changing variables to the $p$-plane ($dD = \frac{dD}{dD} dp$), we have

$$N_w = \frac{1}{2\pi i} \oint_{C_p} \frac{dD}{D} dp$$

3. Representative Mapping

4. Deform contour $C_p$, 

$$N_w = \sum_{j=1}^{n} \frac{1}{2\pi i} \oint_{C_{p_j}} \frac{1}{D} \frac{dD}{dp} dp$$
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T.C. (Continued)  

\[ D(p) = D(p_0) + \left. \frac{\partial D}{\partial p} \right|_{p_0} (p-p_0) + \ldots \]

**Solution**

b. We can also expand the function \( g = \frac{\partial D}{\partial p} \) about \( p = p_0 \):

\[ g(p) = g(p_0) + (p-p_0) \left. \frac{\partial g}{\partial p} \right|_{p_0} + \ldots \]

To lowest order, keep only \( g(p_0) \):

\[ \frac{\partial D}{\partial p} \bigg|_{p_0} = \frac{\partial g}{\partial p} \bigg|_{p_0} = \frac{\partial D}{\partial p} \bigg|_{p_0} \]

c. Thus

\[ \frac{1}{D} \frac{\partial D}{\partial p} = \frac{\partial (g(p_0))}{(p-p_0)(\frac{\partial D}{\partial p})} = \frac{1}{p-p_0} \]

6. Thus we obtain

\[ N_W = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c_{p_0}} \frac{dp}{p-p_0} = n \quad \text{QED.} \]

D. The Penrose Condition

1. Let's apply the Nyquist Criterion to a distribution function with an arbitrary number of humps.

2. We want an expression for \( D(k, \omega) \) valid for any distribution function.

b. Since \( \gamma \) is always small near \( \delta = 0 \), we can use the Plancherel Relation to evaluate \( D_r(k, \omega) \) and \( D_i(k, \omega) \).

(We did this for Weak Gravel Rate Approximation, see #14, II.C.4.)

\[ D_r(k, \omega) = 1 - \frac{c_p^2}{K^2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu - \delta/k} \]

\[ D_i(k, \omega) = -\pi \frac{k}{k^4} \frac{c_p^2}{K^2} \left. \frac{\partial D_r}{\partial \nu} \right|_{\nu = \delta/k} \]
3. Shape of \( i=0 \) curve in \( D \)-plane near \( v_1 = \pm \infty \)
   a. First, we'll assume \( k > 0 \).
   b. Note: \( \frac{\partial F_0}{\partial v_2} \to 0 \) as \( v_2 \to \infty \), so \( \lim_{v_2 \to \infty} D_r = 1, \lim_{v_2 \to \infty} D_i = 0 \).
   c. Also, as \( v_2 \to \infty \) \( \frac{\partial F_0}{\partial v_2} < 0 \) and as \( v_2 \to -\infty \) \( \frac{\partial F_0}{\partial v_2} > 0 \), since \( F_0(v_2) > 0 \) always.
   
   d. Thus

   ![Diagram of \( D \)-plane with \( D_r \), \( D_i \), and \( v_2 \) axes showing behavior at \( v_2 \to \pm \infty \).

4. Crossing \( D_r \)-axis (\( D_i = 0 \))
   a. Contour crosses \( D_r \)-axis when \( D_i = 0 \Rightarrow \frac{\partial F_0}{\partial v_2} = 0 \).
   b. Only crosses \( D_r \) where distribution function has zero slope.
   c. For a smooth, continuous \( F_0(v_2) \), there are always an odd number of crossings.

5. Application of the Winding Theorem
   a. Take contour \( C_p \) downward along \( \sigma = 0 \), closing at infinity in right half-plane.
   b. (Thus \( C_p \) encloses all unstable roots.)
   c. Contour of \( |p| = \infty \) maps to \( D = 1 \).
   d. Residue contour maps to \( \sigma = 0 \) curve.
   e. If \( \sigma = 0 \) curve in \( D \)-space winds around the origin \( (D=0) \) \( (CCW) \), then unstable root exists.
6. For single-hump distribution:
   a. CCW contour $C_D$ must cross Dr-axis to the right of 1 => Stable (Does not include D=0).
   b. Once again, we have proven Gardner's Theorem.

7. Two-humped distribution
   a. Cross Dr-axis three times: Upward at $V_1$ & $V_3$ (maxima) Downward at $V_2$ (minimum)
   b. At the points where $\frac{dF_v}{dv} = 0$, $D_i = 0$ and $V_2 = \frac{0}{k} = V_i$.

Thus, the crossing of the Dr-axis occurs at

$$D_r = 1 - \frac{e^{\frac{-v^2}{k^2}}}{k^2} \int_{-\infty}^{\infty} \frac{dF_v}{dv} \frac{dv}{V_2 - V_i}$$

If $D_r < 0$ at a crossing, the plasma is unstable.

6. The Reverse Condition
   a. Noting that $F_0(V_i) = 0$, we may write

$$\int_{-\infty}^{\infty} \frac{dF_0}{dv} \frac{dv}{V_2 - V_i} = \int_{-\infty}^{\infty} \frac{2}{k} \left[ F_0(V_2) - F_0(V_i) \right]$$
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b. Integrating by parts, we obtain

\[
\int_{-\infty}^{\infty} \frac{F_0(v_2) - F_0(v_1)}{(v_2 - v_1)^2} \, dv_2 > 0
\]

Penrose Condition
for Instability

NOTE: We may drop principal value since numerator = 0 when \( v_2 = v_1 \).

d. Penrose Condition applies for a distribution function with any number of humps.

9. Graphical Interpretation of Penrose Condition

b. Integral is summation of distribution above \( F_0(v_1) \) minus that below \( F_0(v_1) \) weighed by function \( \frac{1}{(v_2 - v_1)^2} \).

\[
\Rightarrow \quad \int_{v_2}^{v_1} \frac{1}{(v_2 - v_1)^2} \, dv_2
\]

c. Thus, humps above minimum of \( F_0(v_1) \) must be large enough that integral is positive.
Lecture 16 (Continued)

1. (Continued) Example:
   E. Counter-Streaming Beam Instability

1. DEFE Counter-Streaming Cauchy Distribution
   \[ F_0(v_z) = \frac{C}{2\pi} \left[ \frac{1}{C^2 + (v_z - V)^2} + \frac{1}{C^2 + (v_z + V)^2} \right] \]

   a. For \( C \to 0 \), \( F_0(v_z) \) goes to two delta functions. The "zero" temperature limit, \textbf{UNSTABLE}.

   b. As \( C \) increases, eventually the distribution transitions to a single hump at \( C = \sqrt{3}V \), \textbf{STABLE} by Gardner's Thm.

   c. Thus, at some point between \( C = 0 \) and \( C = \sqrt{3}V \) the system goes from unstable to stable.

   (For increasing temperature, the distribution becomes stable).

2. Apply Pringle Condition
   a. At peaks at \( v_z = \pm V \), Pringle Condition is clearly negative.
   b. At \( v_z = 0 \), we can show that
   \[ \int_{-\infty}^{\infty} \frac{F_0(v_z) - F_0(0)}{(v_z - 0)^2} = \frac{V^2 - C^2}{(V^2 + C^2)^2} \]

   c. Thus plasma is unstable when \( V^2 - C^2 > 0 \), or \( V > C \).

3. Myer's Criterion
   a. We can also evaluate \( \tilde{D}_r \) and \( \tilde{D}_z \) for this distribution
to show the \( \tilde{D} = 0 \) curve gives an unstable region for \( C < V \).

   Unstable since \( \tilde{D} = 0 \) is inside.
II. Overview of Fluid vs. Kinetic Instabilities

A. Fluid vs. Kinetic Instabilities:
1. The two-stream instability of cold beams is a fluid instability, because all particles react in the same way (no thermal spread of velocities).
2. Such fluid instabilities can be studied by fluid equations.
3. Physical Picture of Two-Stream Instability
   a. Consider a beam of positive charges streaming through a neutralizing background at rest.
   b. Conservation of number density means \( n_i V_i = \text{constant} \).
   c. If a local perturbation leads \( V_i \) to a decrease in beam velocity \( V_i \), at some point, the density of ions must increase.
   d. The resulting electric field leads to a further slowing of the beam. \( \Rightarrow \) Positive feedback \( \Rightarrow \) Unstable
   e. Such an instability will continue to grow until some previously neglected nonlinear term halts the growth.

4. Kinetic Instability
   a. The free energy in a finite temperature collisionless function can lead to growth of instability due to interaction with resonant particles.
   b. Eventually free energy is tapped and kinetic instability occurs.