

29/1/95

Homes ①

Lecture #2: Parallel Propagating Waves & Arbitrary Perpendicular Propagating WavesI. Parallel Propagating Waves ($\mathbf{k} \parallel \mathbf{B}$)A. For $\mathbf{k} \parallel \mathbf{B}$, the angle $\theta = 0$.

1. Solutions are easily found using the "Tangent" form of the dispersion relation

$$\tan^2 \theta = \frac{-P (n^2 - R)(n^2 - L)}{(S n^2 - RL)(n^2 - P)}$$

2. For $\theta = 0$, ~~we have~~ $\tan \theta = 0$, so we must have

a. $P = 0$

b. $n^2 = R$

c. $n^2 = L$

} Three modes for parallel propagation

3. We can understand more about these solutions if we look at the electric field eigenfunction using matrix form of dispersion relation:

a. For $\theta = 0$, we have

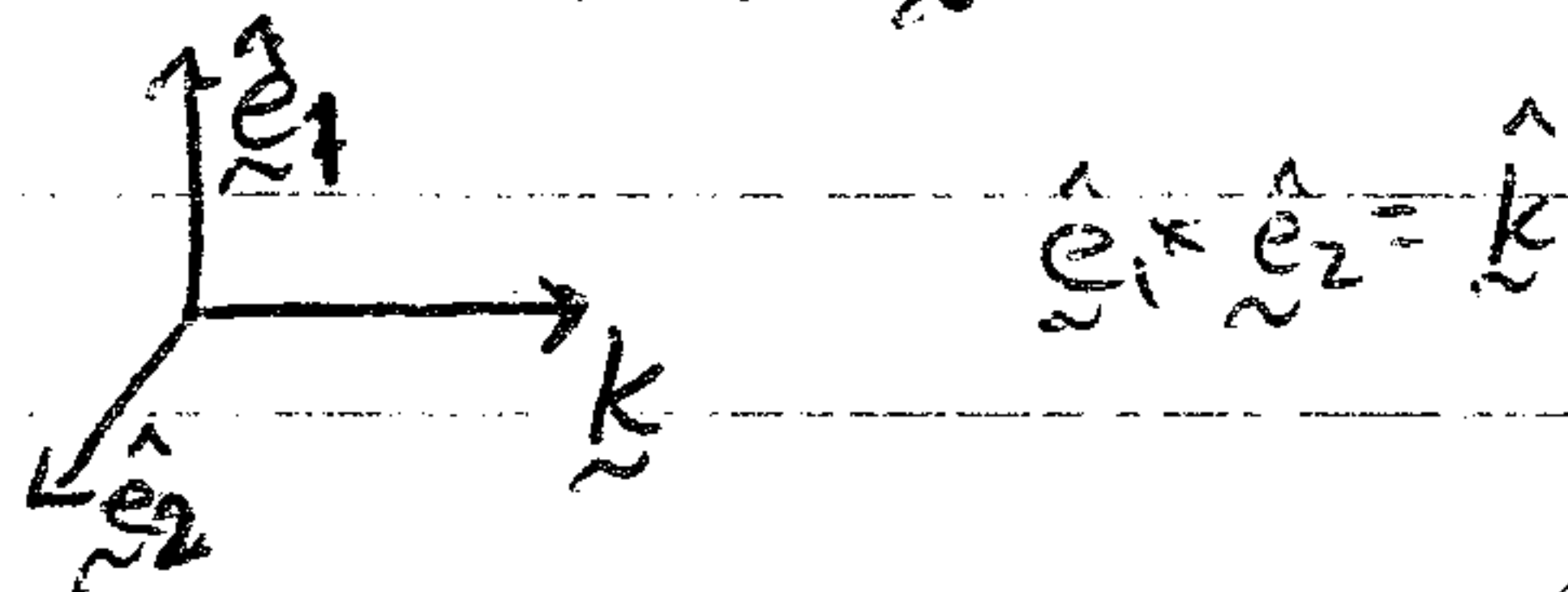
$$\begin{pmatrix} S - n^2 & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

b. Recall that we have performed a Fourier transform, so (E_x, E_y, E_z) are really the Fourier coefficients $\underline{E}(\underline{k})$ such that

$$\underline{E}(\underline{x}, t) = \underline{E}(\underline{k}) e^{i(\underline{k} \cdot \underline{x} - \omega(\underline{k})t)}$$

B. Longitudinal Wave Plasma Oscillation: ($P=0$)1. Recall (from 029/1/94 Lect #22) that longitudinal and transverse are defined with respect to the wave vector direction $\hat{\mathbf{k}}$.

$$\underline{E}_1 = E_{T1} \hat{\mathbf{e}}_1 + E_{T2} \hat{\mathbf{e}}_2 + E_L \hat{\mathbf{k}}$$

2. In a magnetized plasma, the direction of the magnetic field $\hat{\mathbf{b}} = \frac{\mathbf{B}}{|\mathbf{B}|}$ is another important direction.a. For the case of parallel wave vectors ($\mathbf{k} \parallel \hat{\mathbf{b}}$), we have $\hat{\mathbf{e}}_1 = \hat{\mathbf{z}}$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{y}}$

Lecture #2: (Continued)

I B. (Continued)

3. The eigenfunction for the $P=0$ mode has $E_z \neq 0, E_x = E_y = 0.$

$$\Rightarrow \underline{E}_1 = E_L \hat{k}$$

a. This is a longitudinal mode since $\hat{k} \times \underline{E}_1 = 0.$

4. Solution for Frequency

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} = 0$$

$$\Rightarrow \boxed{\omega^2 = \omega_{pi}^2 + \omega_{pe}^2}$$

(remember we assume ion & electron plasma).

a. This is just the usual dispersion relation for the plasma oscillation.

b. This longitudinal mode has motion along the magnetic field.

$$\Rightarrow \text{Lorentz force} \sim \underline{v} \times \underline{B} = 0$$

c. This is just the usual plasma oscillation in the direction along the mean field!

C. Right-hand Polarized Transverse Wave ($n^2 = R$)

1. NOTE: $S - n^2 = S - R = \frac{1}{2}(R+L) - R = -\frac{1}{2}(R-L) = -D.$

Thus, we get
$$\begin{pmatrix} -D & -iD & 0 \\ iD & -D & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

a. Eigenfunction: Take $E_x = E_0$, then $-D(E_0) - iD E_y = 0$
 $\Rightarrow E_y = i E_0$

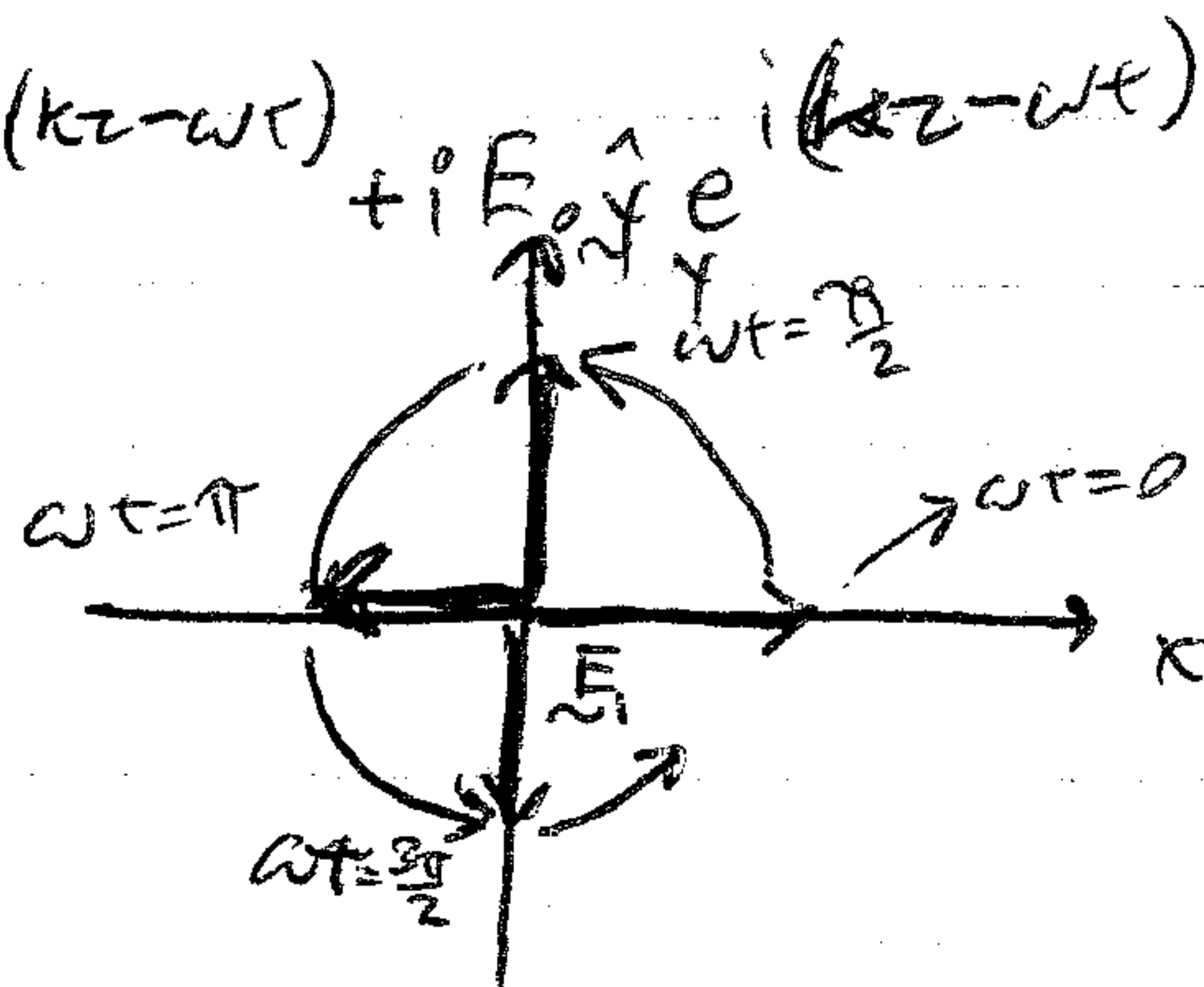
b. For $n^2 = R$, eigenfunction is $\underline{E}_1 = (E_0, i E_0, 0)$

2. Right-handed polarization: $\underline{E}(z,t) = E_0 \hat{x} e^{i(kz - \omega t)} + i E_0 \hat{y} e^{i(kz - \omega t)}$

a. The real component at $z=0$ becomes

circular polarization $\rightarrow E(z=0,t) = E_0 \cos \omega t \hat{x} + E_0 \sin \omega t \hat{y}$

b. This rotates in a counter-clockwise (right-hand) sense with time.



Lecture #2 (Continued)
I.C. (Continued)

Howes ③

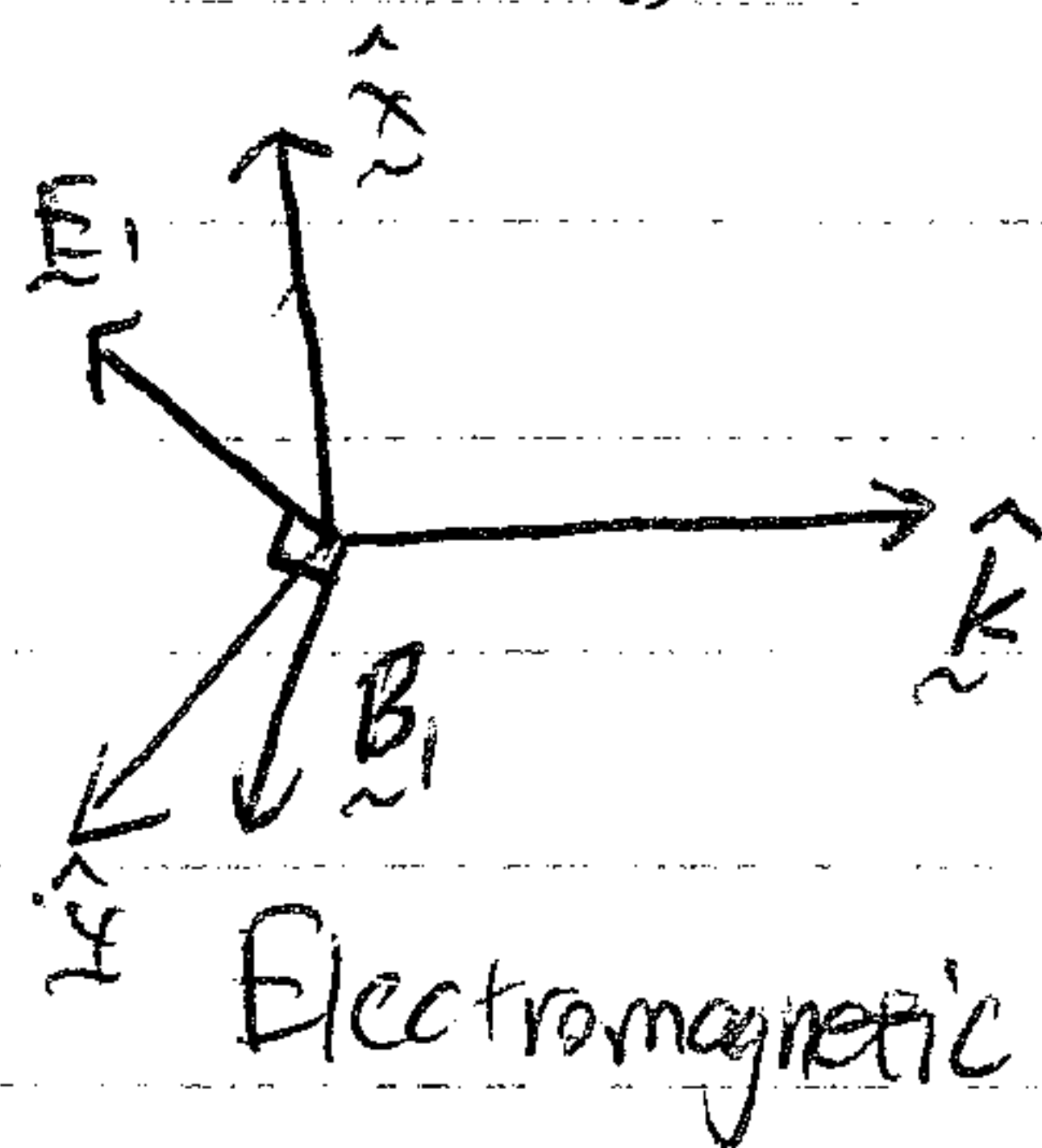
3. Transverse, Electromagnetic Waves

a. Since $\underline{\tilde{E}}_1 = E_0 \hat{x} + i E_0 \hat{y}$, $\underline{\tilde{E}}_1 \times \underline{\tilde{k}} \neq 0 \Rightarrow$ Waves are transverse.

b. Faraday's Law gives $\underline{\tilde{B}}_1 = \frac{\underline{\tilde{k}}}{\omega} \times \underline{\tilde{E}}_1$, so

$$\underline{\tilde{B}}_1 = \frac{\underline{\tilde{k}}}{\omega} \hat{k} \times (E_0 \hat{x} + i E_0 \hat{y}) = \frac{k E_0}{\omega} \hat{y} - \frac{i k E_0}{\omega} \hat{x}$$

$$\Rightarrow \underline{\tilde{E}}_1 = E_0 (\hat{x} + i \hat{y}), \quad \underline{\tilde{B}}_1 = -\frac{i k E_0}{\omega} (\hat{x} + i \hat{y})$$



4. Solution for Frequency: $n^2 = R$

$$a. \frac{c^2 k^2}{\omega^2} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} = 1 - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ce})}$$

where $\omega_{ps}^2 = \frac{n_s q_s^2}{\epsilon_0 m_s}$ and $\omega_{cp}^2 = \omega_{pi}^2 + \omega_{pe}^2 \approx \omega_{pe}^2$

and $\omega_{cs} = \frac{q_s B_0}{m_s}$ (and this carries sign of charge q_s).

b. This solution may be rewritten:

$$\frac{k^2 c^2}{\omega^2} = \frac{(\omega + \omega_L)(\omega - \omega_R)}{(\omega + \omega_{ce})(\omega + \omega_{ci})}$$

where $\omega_L \equiv \frac{\omega_{ce} + \omega_{ci}}{2} + \frac{1}{2} \sqrt{(\omega_{ce} + \omega_{ci})^2 + 4(\omega_{pe}^2 - \omega_{ce}\omega_{ci})}$

$$\omega_R = -\frac{(\omega_{ce} + \omega_{ci})}{2} + \frac{1}{2} \sqrt{(\omega_{ce} + \omega_{ci})^2 + 4(\omega_{pe}^2 - \omega_{ce}\omega_{ci})} = \omega_L - (\omega_{ce}\omega_{ci})$$

NOTE: $\omega_L \omega_R = \omega_{pe}^2 - \omega_{ce}\omega_{ci}$

D. Left-handed Polarized Transverse Waves ($n^2 = L$)

1. Eigenfunction $\underline{\tilde{E}}_1 = (E_0, -i E_0, 0) \Rightarrow$ Left-hand circular polarization

2. Similar to the R solution but with

$$\frac{k^2 c^2}{\omega^2} = \frac{(\omega - \omega_L)(\omega + \omega_R)}{(\omega - \omega_{ce})(\omega - \omega_{ci})}$$

I. (Continued)

E. Limits of R & L waves:

1. NOTE: For most plasmas, $\omega_{ce} < \omega_{pe}$. We'll consider only this case.
 (This corresponds to $\frac{VA^2}{c^2} < \frac{m_e}{m_i}$)

2. Resonances and cutoffs:

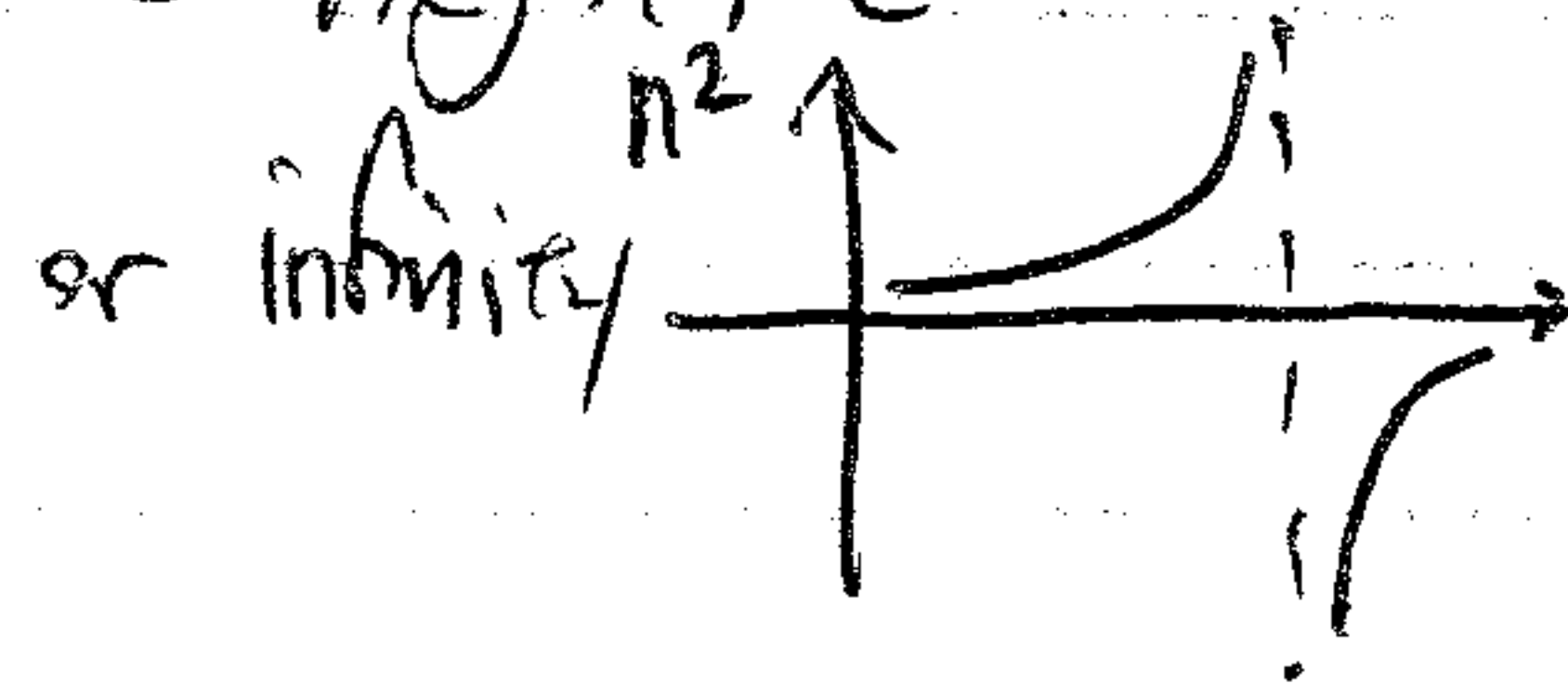
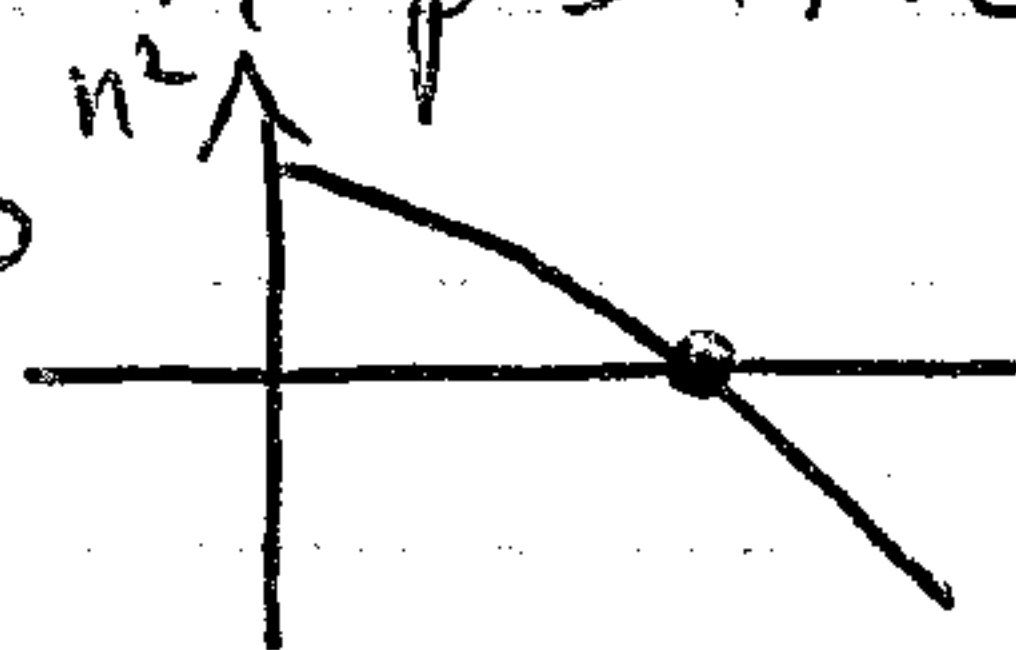
a. Index of refraction $n = \frac{ck}{\omega}$

$$n^2 = \frac{(\omega \pm \omega_L)(\omega \mp \omega_R)}{(\omega \pm \omega_{ce})(\omega \pm \omega_{ci})} \quad \begin{matrix} \text{Signs,} \\ (R) \\ (L) \end{matrix}$$

b. For $n^2 > 0$, we have a propagating wave

$n^2 < 0$, we have an evanescent (damped) solution.

c. Values of n^2 can switch from positive to negative either by going through zero



d. $n^2 \rightarrow \infty \Rightarrow$ Resonance

e. $n^2 \rightarrow 0 \Rightarrow$ Cutoff

3. For R mode, let's simplify using $|\omega_{ce}| \gg \omega_{ci}$ and $\omega_{pe}^2 \gg \omega_{pi}^2$

a. $\Rightarrow \omega_{ce} + \omega_{ci} \approx \omega_{ce}$ and $\omega_{pe}^2 + \omega_{pi}^2 \approx \omega_{pe}^2$

b. Also, let's take absolute values in the formula, $\omega_{ce} = -|\omega_{ce}|$

4. R mode becomes:

$$n^2 = \frac{(\omega + \omega_L)(\omega - \omega_R)}{(\omega - |\omega_{ce}|)(\omega + \omega_{ci})}$$

$$\text{where } \omega_R \approx \frac{|\omega_{ce}|}{2} + \frac{1}{2} \sqrt{\omega_{ce}^2 + 4(\omega_{pe}^2 + |\omega_{ce}|\omega_{ci})}$$

and $\omega_L \approx \omega_R - |\omega_{ce}|$. Thus $\omega_R > \omega_L$

a. $n^2 > 0$ when $\omega > \omega_R$ or $\omega < |\omega_{ce}|$

$n^2 < 0$ (evanescent) when $|\omega_{ce}| < \omega < \omega_R$

Lecture #2 (Continued)

I. E. 4. (Continued)

b. Resonance ($n^2 \rightarrow \infty$) occurs when $\omega = |\omega_{ce}|$.

c. Cutoff ($n^2 \rightarrow 0$) occurs when $\omega = \omega_R$.

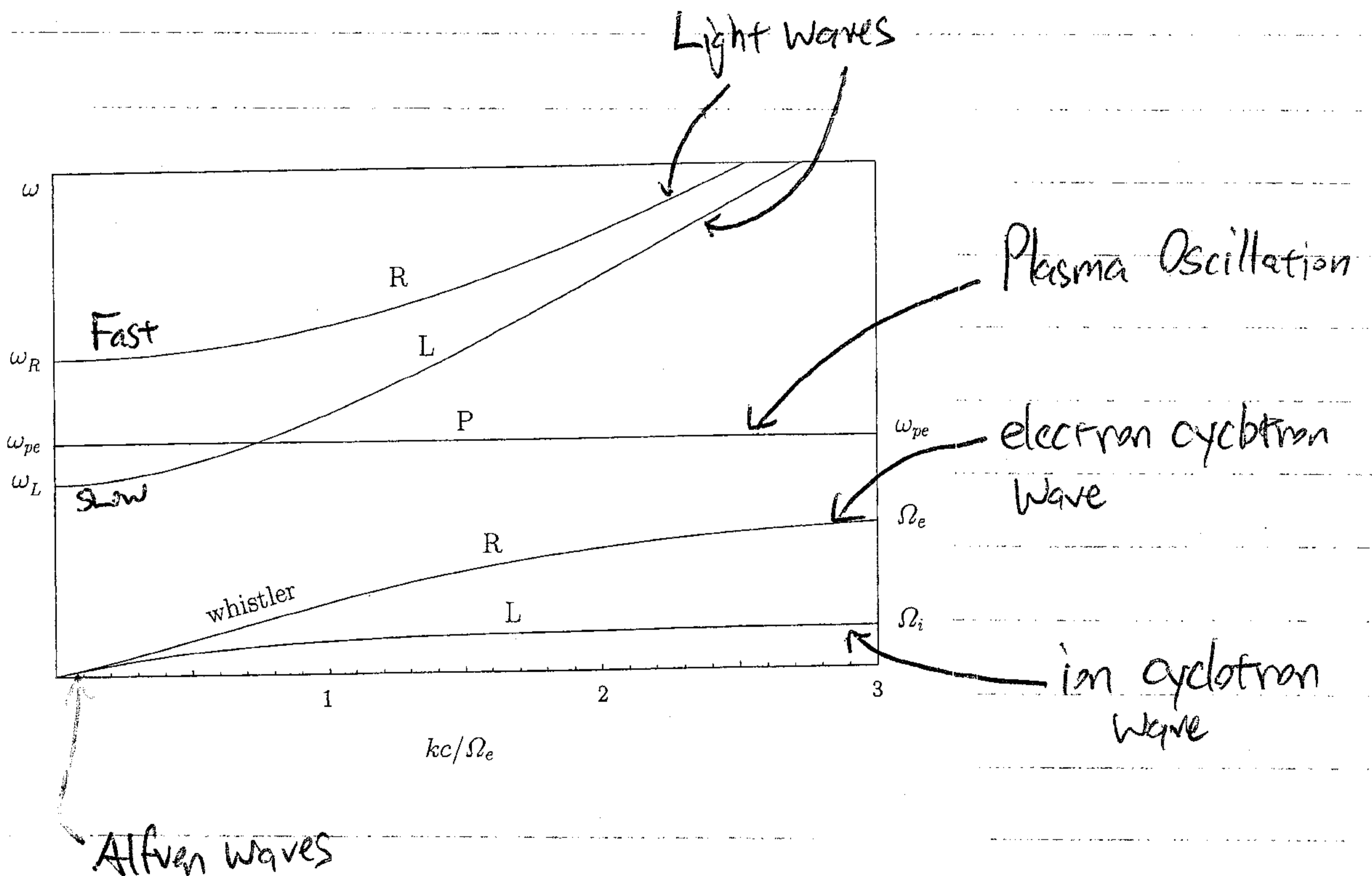
5. For the L mode, a. $n^2 > 0$ when $\omega > \omega_L$ $\omega < \omega_{ci}$

$n^2 < 0$ when $\omega_{ci} < \omega < \omega_L$

b. Resonance ($n^2 \rightarrow \infty$) when $\omega = \omega_{ci}$.

Cutoff ($n^2 \rightarrow 0$) when $\omega = \omega_L$.

6. $\omega(k)$ for modes with parallel wave vector: $\underline{k} = k \hat{z}$.



7. Frequency Limits:

a. Alfvén Waves: Low frequency $\omega \ll \omega_{ci} \ll |\omega_{ce}| < \omega_{pe}$

$$\frac{k^2 c^2}{\omega^2} = \frac{(\omega + \omega_L)(\omega - \omega_R)}{(\omega - |\omega_{ce}|)(\omega + \omega_{ci})} \approx \frac{+\omega_L \omega_R}{+|\omega_{ce}| \omega_{ci}} = \frac{\omega_{pe}^2 \frac{\omega_L \omega_R}{\omega_{ce} \omega_{ci}}}{|\omega_{ce}| \omega_{ci}} = \frac{n_0 e^2}{\epsilon_0 m_e} \frac{q_i B_0}{m_i}$$

$$\Rightarrow \frac{k^2}{\omega^2} \frac{1}{\mu_0 \epsilon_0} = \frac{n_0 m_i}{\epsilon_0 B_0^2} \Rightarrow \boxed{\omega^2 = k^2 V_A^2} \text{ (Same for both L & R modes)}$$

Lecture #2 (Continued)

Pages 6

I. E.7. (Continued)

b. Ion Cyclotron Waves: $\omega \approx \omega_{ci}$ (L mode)

$$\Rightarrow (\omega - \omega_{ci}) \approx \frac{-\omega_{ci}^3}{k^2 v_A^2}$$

c. Whistler Waves: $\omega_{ci} \ll \omega \ll |\omega_{ce}|$ (R mode)

$$\Rightarrow \omega = \frac{k^2 c^2}{\omega_{pe}^2} |\omega_{ce}|$$

d. Electron Cyclotron Waves: $\omega \approx |\omega_{ce}|$ (R mode)

$$\Rightarrow (\omega - |\omega_{ce}|) \approx \frac{|\omega_{ce}|}{\omega_{pe}^2} \left(1 - \frac{k^2 c^2}{\omega^2}\right)^{-1}$$

e. Left and Right-hand polarized light waves (R & L modes)

1. $\omega > \omega_R$ or $\omega > \omega_L \Rightarrow |\omega_{ce}| \gg \omega_{ci}$

2. ~~These~~ For R mode:

$$\frac{k^2 c^2}{\omega^2} = \frac{(\omega + \omega_L)(\omega - \omega_R)}{(\omega + |\omega_{ce}|)(\omega + \omega_{ci})} \approx \frac{\omega^2 + \omega(\omega_L - \omega_R) + \omega_L \omega_R}{\omega^2}$$

$$\Rightarrow k^2 c^2 = \omega^2 + \omega(|\omega_{ce}| - \omega_{pe}^2 + |\omega_{ce}| \omega_{ci}) \Rightarrow \boxed{\omega^2 \approx c^2 k^2 + \omega_{pe}^2}$$

Usual modified light wave dispersion relation.

II. Perpendicular propagating waves ($\underline{k} \perp \underline{B}$)

A.1. For $\underline{k} \perp \underline{B}$, we have $\theta = \frac{\pi}{2}$ or $\tan \theta \rightarrow \infty$

2. Solutions occurs for

a. $n^2 = P$

b. $n^2 = \frac{RL}{S}$

} Two modes for perpendicular propagation.

Lecture #2 (Continued)

Haves ⑦

II. A. (Continued)

3. For $\theta = \frac{\pi}{2}$,
Eigenfunctions are given by

$$\begin{pmatrix} S & -iD & 0 \\ iD & S-n^2 & 0 \\ 0 & 0 & P-n^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

B. Ordinary Mode ($n^2 = P$)

1. For the solution $n^2 = P$, the eigenfunction is $\underline{E}_1 = (0, 0, E_0)$

2. Transverse Wave:

a. Now we have $\underline{k} = k \hat{z}$, so $\underline{k} \times \underline{E}_1 \neq 0 \Rightarrow$ Transverse

b. Likewise, this is an electromagnetic wave.

3. Solution for frequency: $n^2 = P$

a. $\frac{k^2 c^2}{\omega^2} = 1 - \sum_s \frac{\omega_p s^2}{\omega^2} \Rightarrow k^2 c^2 = \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2)$

$\Rightarrow \boxed{\omega^2 = k^2 c^2 + \omega_{pi}^2}$ The usual Modified light wave.

b. Moments due to the electric field are parallel to the magnetic field, \Rightarrow there is no $\underline{v} \times \underline{B}$ Lorentz force.

\Rightarrow The magnetic field has no effect on waves of this polarization (parallel to \underline{B}).

$\Rightarrow \boxed{\text{Ordinary Mode}}$ (linearly polarized).