Lecture #20 Quasi-Linear Theory, Part II

I. Review:

A. Last time, we calculated the Quasilinear Equations

1. $$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}\right) f_{s1} = \frac{q_s}{m_s} \frac{\partial \phi_1}{\partial z} \frac{\partial \langle f_s \rangle}{\partial v_z}$$

2. $$\frac{\partial^2 \phi_1}{\partial z^2} = -\sum_s \frac{q_s}{\epsilon_0} \int_{-\infty}^{\infty} dv_z \, f_{s1}$$

$\Big\}\leftarrow$ Fast evolution of Fluctuations

3. $$\frac{\partial \langle f_s \rangle}{\partial \tau} = \frac{q_s}{m_s} \frac{\partial}{\partial v_z} \left\langle f_{s1} \frac{\partial \phi_1}{\partial z} \right\rangle$$

$\Big\}\rightarrow$ Slow evolution of Mean

B. Fourier Transforms in Space & Time of ϕ_1 & f_{s1}

1. Space: $$\phi_1(z,t) = \int_{-\infty}^{\infty} dk \, \tilde{\phi}_1(k,t) e^{ikz}$$

$$f_{s1}(z,v_z,t) = \int_{-\infty}^{\infty} dk \, \tilde{f}_{s1}(k,v_z,t) e^{ikz}$$

2. Time: $$\tilde{\phi}_1(k,t) = \hat{\phi}_1(k) e^{-i\omega(k,\tau)t}$$

$$\tilde{f}_{s1}(k,v_z,t) = \hat{f}_{s1}(k,v_z) e^{-i\omega(k,\tau)t}$$

where frequency $\omega(k,\tau)$ may change on long timescale $\tau \equiv t/\epsilon^2$

C. Reality Condition
1. $\hat{\phi}_1(k) = \hat{\phi}_1^*(-k)$
2. $\omega_r(k,\tau) = -\omega_r(-k,\tau)$
   $\gamma(k,\tau) = \gamma(-k,\tau)$

II. Derivation of Quasilinear Diffusion Equation (Continued)

A. Evolution of Mean Distribution

1. $$\frac{\partial \langle f_s \rangle}{\partial t} = \frac{q_s}{m_s} \frac{\partial}{\partial v_z} \underbrace{\left\langle f_{s1} \frac{\partial \phi_1}{\partial z} \right\rangle}$$

Let's calculate this using solutions for f_{s1} & ϕ_1.

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Howes 3

II.A. (Continued)

2. $\frac{\partial}{\partial v_z}\left\langle f_{s1} \frac{\partial \phi_1}{\partial z}\right\rangle = \frac{\partial}{\partial v_z} \frac{1}{2L}\int_{-L}^{L} dz\, f_{s1} \frac{\partial \phi_1}{\partial z}$

3. Substituting in Spatial Fourier Transform $\tilde{f}_{s1}$ and $\tilde{\phi}_1$,

$$= \frac{\partial}{\partial v_z}\frac{1}{2L}\int_{-L}^{L} dz \left[\int_{-\infty}^{\infty} dk\, \tilde{f}_{s1}(k,v_z,t)\, e^{ikz}\right] \underbrace{\left\{\frac{\partial}{\partial z}\left[\int_{-\infty}^{\infty} dk'\, \tilde{\phi}_1(k',t) e^{ik'z}\right]\right\}}_{=\int_{-\infty}^{\infty} dk'\, ik'\, \tilde{\phi}_1(k',t) e^{ik'z}}$$

4. Collecting all terms dependent on z:

$$= \frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \frac{ik'}{2L} \tilde{f}_{s1}(k,v_z,t)\, \tilde{\phi}_1(k',t) \left[\int_{-L}^{L} dz\, e^{i(k+k')z}\right]$$

5. NOTE: Identity: $\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x-x_0)}\, d\omega$

a. Thus, in the limit of large box size $L \to \infty$ (necessary for chosen boundary conditions), we have

$$\lim_{L\to\infty}\int_{-L}^{L} dz\, e^{i(k+k')z} = 2\pi\,\delta(k+k')$$

6. The δ-function can be used to evaluate the k' integral at $k' = -k$:

$$= \frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} dk \frac{2\pi}{2L}(-ik)\tilde{f}_{s1}(k,v_z,t)\,\tilde{\phi}_1(-k,t) = \frac{-\pi}{L}\frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} dk\, ik\,\tilde{\phi}_1(-k,t)\,\tilde{f}_{s1}(k,v_z,t)$$

7. Now, let's substitute Time Fourier Transform $\hat{\phi}_1$ & $\hat{f}_{s1}$

a. $$= \frac{-\pi}{L}\frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} dk\, ik \left[\hat{\phi}_1(-k) e^{-i\omega(-k,t)t}\right]\left[\hat{f}_{s1}(k,v_z) e^{-i\omega(k,t)t}\right]$$

Lecture #20 (Continued) Howes ③

II. A.7. (Continued)

b. $= -\frac{\pi}{L}\frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} dk \; ik\,\hat{\phi}_1(-k)\hat{f}_{s1}(k,v_z)\, e^{-i[\omega(-k,\tau)+\omega(k,\tau)]t}$

8. NOTE: $\omega(-k,\tau)+\omega(k,\tau) = -\cancel{\omega_r(k,\tau)}+i\gamma(k,\tau)+\cancel{\omega_r(k,\tau)}+i\gamma(k,\tau)$
$= 2i\gamma(k,\tau)$

9. Thus $\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \phi_1}{\partial z}\right\rangle = -\frac{\pi}{L}\frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} dk \; ik\,\hat{\phi}_1(-k)\hat{f}_{s1}(k,v_z)\, e^{+2\gamma(k,\tau)t}$

10. Now, we'll substitute the solution for $\hat{f}_{s1} = -\frac{q_s}{m_s}\frac{k\hat{\phi}_1(k)}{(\omega-kv_z)}\frac{\partial\langle f_s\rangle}{\partial v_z}$
(see III C.3. of Lect #19)

$$= +\frac{\pi q_s}{L m_s}\frac{\partial}{\partial v_z}\int_{-\infty}^{\infty} dk \; ik^2\frac{\hat{\phi}_1(-k)\hat{\phi}_1(k)}{\omega-kv_z}\underbrace{\frac{\partial\langle f_s\rangle}{\partial v_z}}_{\text{NOTE: Does not depend on } k.} e^{2\gamma(k,\tau)t}$$

11. We can use $\hat{E}_1(k) = -ik\hat{\phi}_1(k)$ to eliminate $\hat{\phi}_1$:

$$= \frac{\pi}{L}\frac{q_s}{m_s}\frac{\partial}{\partial v_z}\left\{\left[\int_{-\infty}^{\infty} dk \; \frac{i\,\hat{E}_1(k)\hat{E}_1(-k)}{\omega-kv_z}e^{2\gamma(k,\tau)t}\right]\frac{\partial\langle f_s\rangle}{\partial v_z}\right\}$$

B. Writing in terms of Spectral Energy Density of Electric Field

1. The Electrostatic energy density is given by

$$W_E = \frac{\epsilon_0|E_1(z,t)|^2}{2}$$

a. Averaging this energy density gives $\langle W_E\rangle = \frac{\epsilon_0}{4L}\int_{-L}^{L} dz\,|E_1(z,t)|^2$

Lecture #20 (Continued)

III B. (Continued)

2. We can write $\langle W_E \rangle$ in terms of $\tilde{E}_1(k,t)$ by

$$\langle W_E \rangle = \frac{\epsilon_0}{4L}\int_{-L}^{L} dz \left[\int_{-\infty}^{\infty} dk\, \tilde{E}_1(k,t) e^{ikz}\right]\left[\int_{-\infty}^{\infty} dk'\, \tilde{E}_1^*(k',t) e^{-ik'z}\right]$$

$$= \frac{\epsilon_0}{4L}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'\, \tilde{E}_1(k,t)\, \tilde{E}_1^*(k',t) \underbrace{\int_{-L}^{L} dz\, e^{i(k-k')z}}_{=2\pi\,\delta(k-k')}$$

$$= \frac{\pi\epsilon_0}{2L}\int_{-\infty}^{\infty} dk\, \tilde{E}_1(k,t)\, \tilde{E}_1^*(k,t)$$

3. Thus $$\langle W_E \rangle = \int_{-\infty}^{\infty} dk\, \mathcal{E}(k,t)$$

where we define the Spectral Energy Density $\boxed{\mathcal{E}(k,t) = \frac{\pi\epsilon_0}{2L}|\tilde{E}(k,t)|^2}$

4. We may write the time dependence of $\mathcal{E}(k,t)$ as

a. $$\mathcal{E}(k,t) = \frac{\pi\epsilon_0}{2L}|\hat{E}_1(k)|^2 e^{2\gamma(k,\tau)t}$$

b. This implies $$\frac{d\mathcal{E}(k,t)}{dt} = 2\gamma(k,\tau)\,\mathcal{E}(k,t)$$

τ is treated as a constant

C. Putting it all together

1. We have $$\frac{\partial\langle f_s\rangle}{\partial t} = \frac{\cancel{\pi} q_s^2}{\cancel{L} m_s^2}\frac{\partial}{\partial v_z}\left\{\left[\int_{-\infty}^{\infty} dk\, \frac{i\frac{2\cancel{L}}{\cancel{\pi}\epsilon_0}\mathcal{E}(k,t)}{\omega - kv_z}\right]\frac{\partial\langle f_s\rangle}{\partial v_z}\right\}$$

where we note $\hat{E}_1(k)\,\hat{E}_1(-k) = \hat{E}_1(k)\,\hat{E}_1^*(k) = |\hat{E}(k)|^2$

Lecture #20 (Continued)

III C. (Continued)

2. Finally we obtain

$$\boxed{\frac{\partial \langle f_s \rangle}{\partial t} = \frac{\partial}{\partial v_z}\left[D_q(v_z,t) \frac{\partial \langle f_s \rangle}{\partial v_z} \right]}$$

Quasilinear Diffusion Equation

where the Quasilinear Diffusion Coefficient is

$$\boxed{D_q(v_z,t) = \frac{2}{\epsilon_0}\left(\frac{q_s}{m_s}\right)^2 \int_{-\infty}^{\infty} dk \; \frac{i\mathcal{E}(k,t)}{\omega - k v_z}}$$

D. Reality of Quasilinear Diffusion Coefficient

1. NOTE: $\frac{i\mathcal{E}(k,t)}{\omega - k v_z} = \frac{i\mathcal{E}(\omega_r - k v_z - i\gamma)}{(\omega_r + i\gamma - k v_z)(\omega_r - i\gamma - k v_z)}$

$$= \frac{i\mathcal{E}(\omega_r - k v_z) + \mathcal{E}\gamma}{(\omega_r - k v_z)^2 + \gamma^2}$$

2. $$\int_{-\infty}^{\infty} dk \; \frac{i \overbrace{\mathcal{E}(k,t)}^{\text{Even}} \overbrace{(\omega_r(k,\tau) - k v_z)}^{\text{odd}} + \overbrace{\mathcal{E}(k,t)}^{\text{Even}} \overbrace{\gamma(k,\tau)}^{\text{Even}}}{\underbrace{(\omega_r(k,\tau) - k v_z)^2}_{(\text{odd})^2 = \text{even}} + \underbrace{\gamma^2(k,\tau)}_{\text{even}}}$$

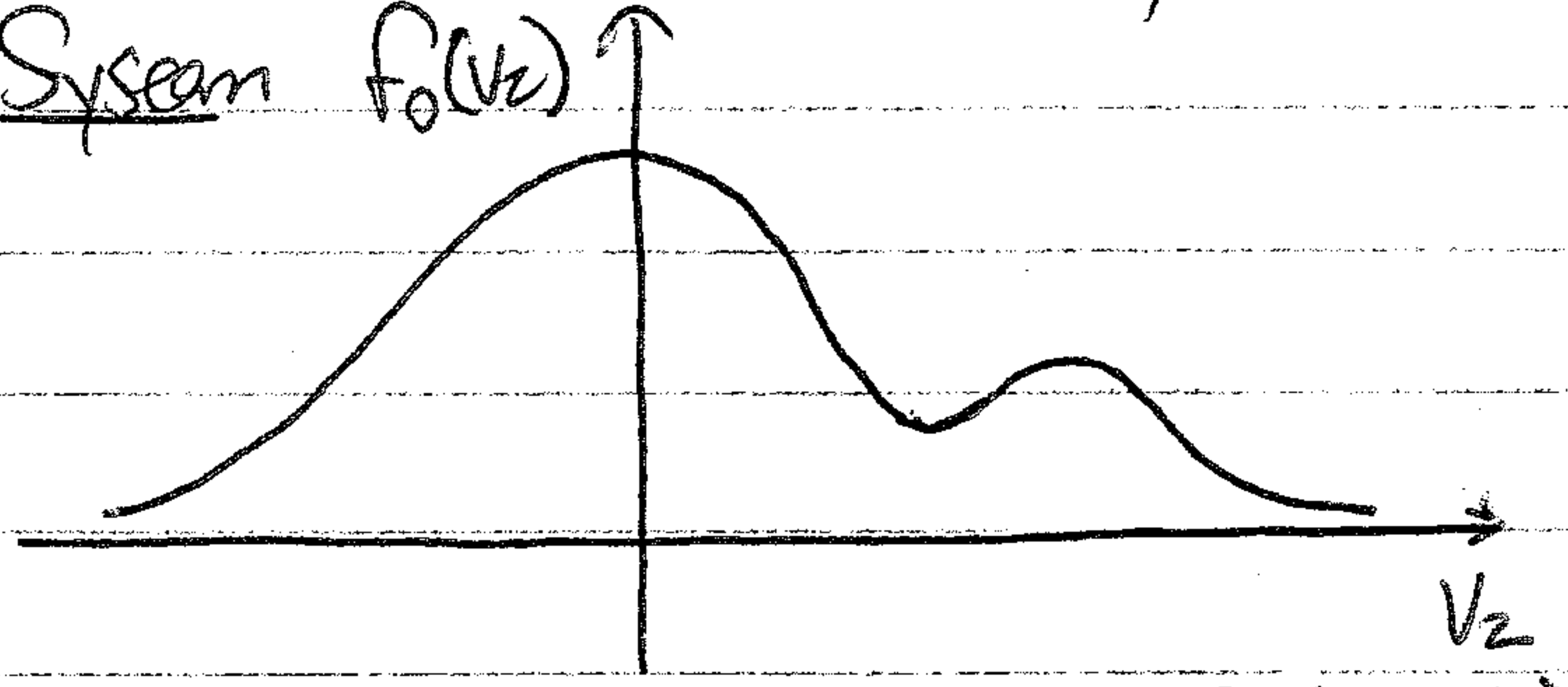
a. Since first term is overall odd, it cancels on integration over $\int_{-\infty}^{\infty} dk$.

b. Thus, we are left with a real quantity:

$$D_q(k_z,t) = \frac{2}{\epsilon_0}\left(\frac{q_s}{m_s}\right)^2 \int_{-\infty}^{\infty} dk \; \frac{\mathcal{E}(k,t)\gamma(k,\tau)}{(\omega_r(k,\tau) - k v_z)^2 + \gamma^2(k,\tau)}$$

III. Application of Quasilinear Theory:

A. 1. Initial System $f_0(v_z)$



2. Three Quantities must be evolved: a. $\langle f_s(v_z,\tau)\rangle$ Mean Distribution

b. $\mathcal{E}(k,t,\tau)$ Spectral Energy Density

c. $\gamma(k,\tau)$ Growth Rate

3. Evolution Equations:

a. $$\frac{\partial \langle f_s\rangle}{\partial \tau} = \frac{\partial}{\partial v_z}\left[D_q(v_z,t,\tau)\frac{\partial\langle f_s\rangle}{\partial v_z}\right]$$

where $$D_q(v_z,t,\tau) = \frac{2}{\epsilon_0}\left(\frac{q_s}{m_s}\right)^2\int_{-\infty}^{\infty} dk\, \frac{\mathcal{E}(k,t,\tau)\,\gamma(k,\tau)}{(\omega_r(k,\tau)-kv_z)^2+\gamma^2(k,\tau)}$$

b. $$\frac{\partial \mathcal{E}(k,t,\tau)}{\partial t} = 2\gamma(k,\tau)\,\mathcal{E}(k,t,\tau)$$

c. $$D(k,\omega) = 1-\sum_s \frac{\omega_{ps}^2}{k^2}\int_{-\infty}^{\infty} dv_z\,\frac{\partial\langle f_s\rangle/\partial v_z}{v_z-\frac{\omega}{k}} = 0$$

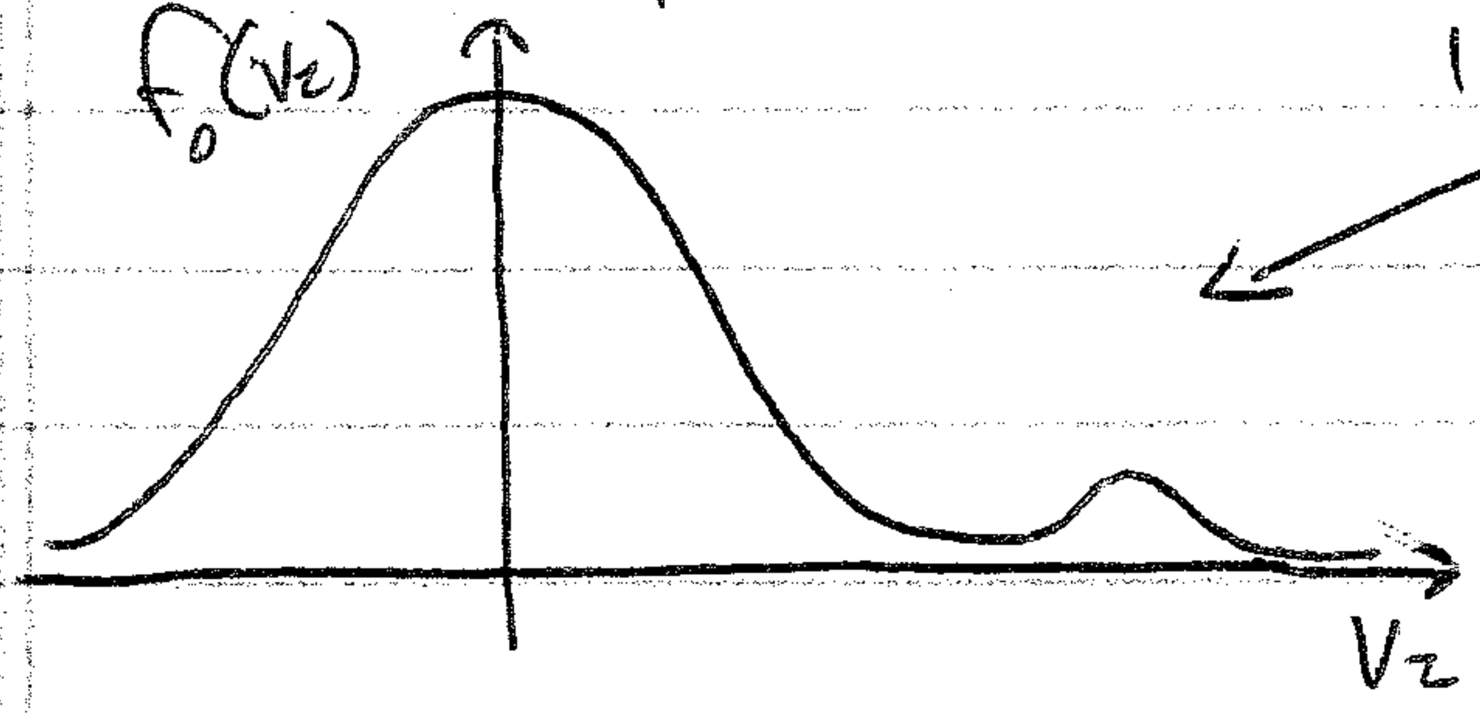
The solution of the dispersion relation yield $\gamma(k,\tau)$ for unstable modes.

$\boxed{\text{Thus, } \langle f_s(v_z,\tau)\rangle,\ \mathcal{E}(k,t,\tau),\text{ and } \gamma(k,\tau) \text{ must be advanced in time self-consistently.}}$

Lecture #20 (Continued)

III. (Continued)

B. The Bump-on-Tail Instability



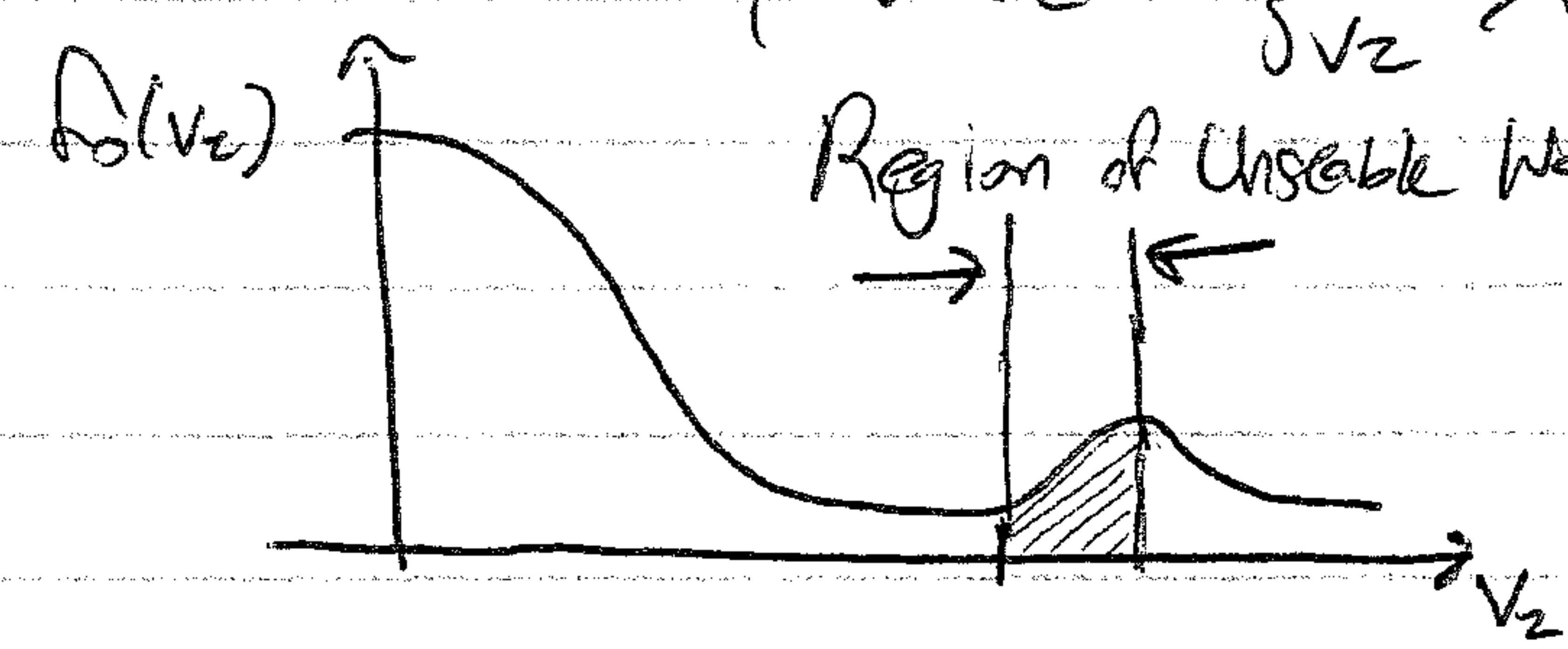
1. Initial Distribution

2. Assume $|\gamma| \ll |\omega_r|$
$\Rightarrow$ Weak Growth Rate:

3. Remember from lect #14 II.C.5., in the weak growth approximation,

a. $$\gamma = \pi \frac{k}{|k|} \frac{1}{\partial D_r / \partial \omega_r} \sum_s \frac{\omega_{ps}^2}{k^2} \left. \frac{\partial F_{s0}}{\partial v_z} \right|_{v_z = \frac{\omega}{k}}$$

b. Thus $\gamma > 0$ only where $\frac{\partial F_{s0}}{\partial v_z} > 0$ (for $v_z > 0$)



4. Since $\frac{\partial \mathcal{E}(k,t,\tau)}{\partial t} = 2\gamma(k,\tau)\mathcal{E}(k,t,\tau)$, the spectral energy density $\mathcal{E}(k)$ only grows in this range of unstable waves with $v_z = \frac{\omega}{k}$.

5. a. For a small, unstable growth rate $|\gamma| \ll |\omega_r|$, we can estimate $$\frac{\gamma}{(\omega_r - kv_z)^2 + \gamma^2} \simeq \pi \delta(\omega_r - kv_z)$$

b. We can thus evaluate $D_q(v_z, t, \tau)$

$$D_q(v_z,t,\tau) = \frac{2}{\epsilon_0}\left(\frac{q_s}{m_s}\right)^2 \int_{-\infty}^{\infty} dk\, \mathcal{E}(k,t,\tau)\, \pi\, \delta(\omega_r - kv_z) = \frac{2\pi}{\epsilon_0}\left(\frac{q_s}{m_s}\right)^2 \mathcal{E}\left(k = \frac{\omega_r}{v_z}\right)$$

Diffusion due to resonant waves

Lecture #20 (Continued) Howes (8)

III. B. (Continued)

6. Therefore

a. $$\frac{\partial \langle f_s \rangle}{\partial t}(v_z, \tau) = \frac{\partial}{\partial v_z}\left\{\frac{2\pi}{\epsilon_0}\left(\frac{q_s}{m_s}\right)^2 \mathcal{E}\left(k = \frac{\omega_r}{v_z}\right) \frac{\partial \langle f_s \rangle}{\partial v_z}\right\}$$

For each v_z, only unstable wave solutions of Dispersion relation with $v_z = \frac{\omega_r}{k}$ lead to diffusion.

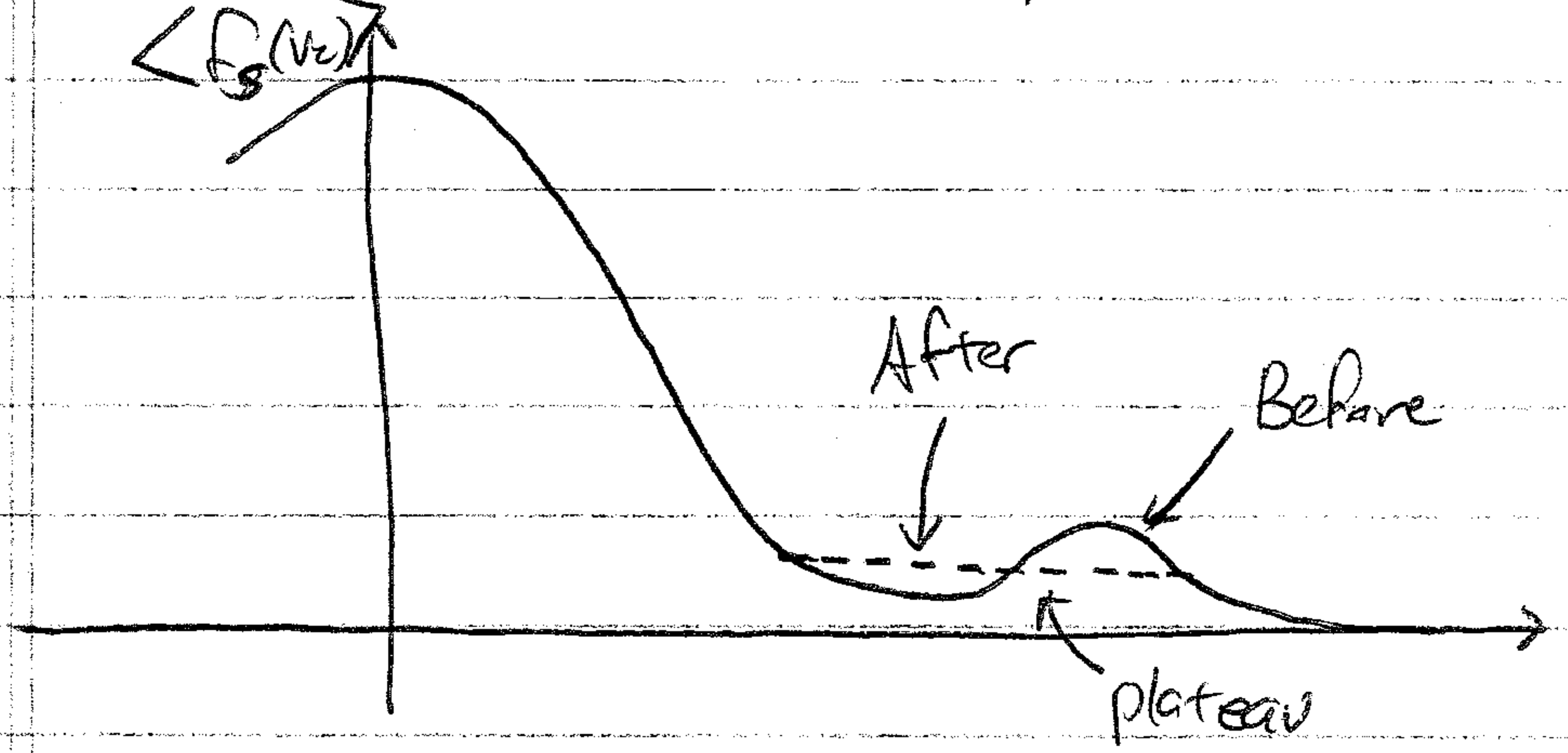
b. The equation is of the form: $\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_z}\left[K \frac{\partial f}{\partial v_z}\right] \sim K \frac{\partial^2 f}{\partial v_z^2}$

Thus, diffusion in Velocity Space!

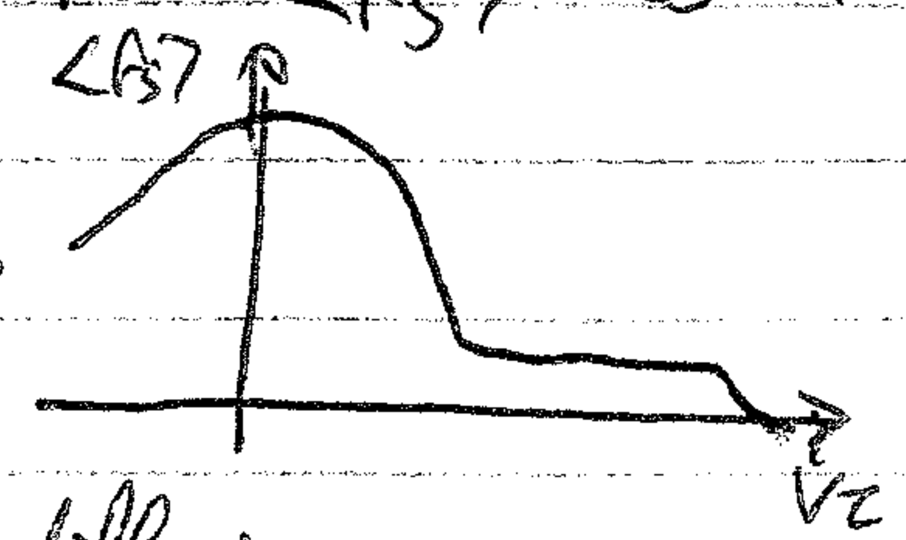
c. This diffusion serves to smooth out the distribution function in velocity space in regions where unstable waves exist, i.e. $\frac{\partial f}{\partial v_z} > 0$.

7. The Result

a. Since particles must be conserved, area under curve must not change.



b. Since diffusion occurs only in regions where $\frac{\partial \langle f_s \rangle}{\partial v_z} > 0$, the effect of quasilinear diffusion is to evolve $\langle f_s \rangle$ to a state where $\frac{\partial \langle f_s \rangle}{\partial v_z} \geq 0$ everywhere

$\Rightarrow$ 

c. At this point you reach marginal stability, unstable waves are no longer generated, and quasilinear diffusion stops.