I. Review:
A. Last time, we calculated the Quasilinear Equations:

1. \[ \left( \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} \right) f_{s1} = \frac{q_s}{m_s} \frac{\partial}{\partial v_z} \frac{\partial}{\partial v_z} \langle f_{s1} \frac{\partial \phi_1}{\partial z} \rangle \rightarrow \text{Fast evolution of fluctuations} \]

2. \[ \frac{\partial^2 \phi_1}{\partial z^2} = -\frac{q_s}{\epsilon_0} \int_{-\infty}^{\infty} dv_z f_{s1} \]

3. \[ \frac{\partial}{\partial t} \langle f_{s1} \frac{\partial \phi_1}{\partial z} \rangle \rightarrow \text{Slow evolution of Mean} \]

B. Fourier Transforms in Space & Time of \( \phi_1 \) & \( f_{s1} \)

1. Space: \[ \hat{\phi}_1(k, t) = \int_{-\infty}^{\infty} dk \hat{f}_{s1}(k, t) e^{ikz} \]

2. Time: \[ \hat{f}_{s1}(k, v_z, t) = \hat{f}_{s1}(k, v_z, t) e^{-i\omega(k, t) \tau} \]

where frequency \( \omega(k, t) \) may change on long timescale \( T \equiv \tau / \epsilon^2 \)

C. Reality Condition

1. \( \hat{\phi}_1(k) = \hat{\phi}_1^*(-k) \)

2. \( \omega_f(k, \tau) = -\omega_f(-k, \tau) \)

\[ \gamma(k, \tau) = \gamma(-k, \tau) \]

II. Derivation of Quasilinear Diffusion Equation (Continued)

A. Evolution of Mean Distribution

1. \[ \frac{\partial}{\partial t} \langle f_{s1} \frac{\partial \phi_1}{\partial z} \rangle \rightarrow \text{Let's calculate this using} \]

Solutions for \( f_{s1} \) & \( \phi_1 \)
Lecture #20 (Continued)
II.A. (Continued)

\[ \frac{\partial}{\partial z} \langle f_{s_1} \phi_1 \rangle = \frac{2}{\sqrt{2}} \int_{L}^{L} \frac{dz}{2L} f_{s_1} \frac{\partial \phi_1}{\partial z} \]

3. Substituting in Spiegel Fourier Transforms \( \hat{f}_{s_1} \) and \( \hat{\phi}_1 \),

\[ = \frac{2}{\sqrt{2}} \int_{L}^{L} \frac{dz}{2L} \left[ \int_{-\infty}^{\infty} \hat{f}_{s_1}(k,v) e^{ikz} dk \right] \left\{ \frac{2}{\sqrt{2}} \int_{-\infty}^{\infty} \hat{\phi}_1(k,v) e^{ikz} dk \right\} \]

4. Collecting all terms dependent on \( z \):

\[ \hat{f}_{s_1}(k,v) \hat{\phi}_1(k,v) \]

\[ = \frac{2}{\sqrt{2}} \int_{L}^{L} \frac{dz}{2L} \left[ \int_{-\infty}^{\infty} i(k+k')z \right] \]

5. Note: Identity: \( \delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwx-x0} dw \)

Thus, in the limit of large box size \( L \to \infty \) (necessary for chosen boundary conditions), we have

\[ \lim_{L \to \infty} \int_{-L}^{L} \frac{dz}{2L} e^{-i(k+k')z} = 2\pi \delta(k+k') \]

6. The \( \delta \)-function can be used to evaluate the \( k' \) integral at \( k' = -k \):

\[ = \frac{2}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{2\pi}{2L} (-ik) \hat{f}_{s_1}(k,v) \hat{\phi}_1(k,v) = -\frac{\pi}{2} \frac{2}{\sqrt{2}} \int_{-\infty}^{\infty} dk ik \hat{\phi}_1(-k,v) \hat{f}_{s_1}(k,v) \]

7. Now, let's substitute Time Fourier Transform \( \hat{\phi}_1 \) & \( \hat{f}_{s_1} \)

\[ a_1 = -\frac{\pi}{2} \frac{2}{\sqrt{2}} \int_{-\infty}^{\infty} dk ik \left[ \hat{\phi}_1(k) e^{-i\omega(k,v)+} \right] \left[ \hat{f}_{s_1}(k,v) e^{-i\omega(k,v)+} \right] \]
b. \[ b_1 = -\frac{\pi}{2} \frac{2}{\nu z} \int_{-\infty}^{\infty} \, dk \, ik \, \hat{\phi}_1(-k) \hat{\rho}_{s_1}(k, \nu z) e^{-i[\omega(-k, \nu z) + \omega(k, \nu z)]} \]

8. **NOTE!** \[ \omega(-k, \nu z) + \omega(k, \nu z) = -\omega(k, \nu z) + i \nu z + \omega(k, \nu z) + i \nu z \]

\[ = 2i \nu z(k, \nu z) \]

9. Thus \[ \frac{2}{\nu z} \, \left( \phi_{s_1} \frac{\partial \hat{\phi}_1}{\partial s_1} \right) = -\frac{\pi}{2} \frac{2}{\nu z} \int_{-\infty}^{\infty} \, dk \, ik \, \hat{\phi}_1(k) \hat{\rho}_{s_1}(k, \nu z) e^{-i \nu z(k, \nu z)} \]

10. Now, we'll substitute the solution for \[ \hat{\rho}_{s_1} = -\frac{q_s}{m_s} \frac{k \hat{\phi}_1(k)}{\omega - k\nu z} \]

\[ \left( \text{see III.C.3. and lecture 13} \right) \]

\[ = \frac{q_s}{2mvz} \int_{-\infty}^{\infty} \, dk \, \frac{k^2 \hat{\phi}_1(-k) \hat{\phi}_1(k)}{\omega - k\nu z} \frac{\partial^2 \hat{\phi}_1}{\partial s_1^2} e^{2i \nu z(k, \nu z)} \]

**NOTE:** Does not depend on \( k \).

11. We can use \( \hat{E}_1(k) = -ik \hat{\phi}_1(k) \) to eliminate \( \hat{\rho}_{s_1} \):

\[ \hat{E}_1(k) = \frac{q_s}{2m} \int_{-\infty}^{\infty} \, dk \, \frac{i \hat{E}_1(k) \hat{E}_1(-k)}{\omega - k\nu z} e^{2i \nu z(k, \nu z)} \]

B. Writing in terms of Spectral Energy Density of Electric Field

1. The Electrostatic energy density is given by

\[ W_E = \frac{\varepsilon_0 |E(z, t)|^2}{2} \]

2. Averaging this energy density gives \( \langle W_E \rangle = \frac{\varepsilon_0}{4\pi} \int d^2 l |E(z, t)|^2 \)
2. We can write \(\langle W_E \rangle\) in terms of \(\hat{E}_1(k,t)\) by
\[
\langle W_E \rangle = \frac{E_0}{4L} \int_{-L}^{L} dz \left[ \int_{0}^{\infty} \hat{E}_1(k,t) e^{ikz} \right] \left[ \int_{0}^{\infty} \hat{E}_1^*(k,t) e^{-ikz} \right]
\]
\[
= \frac{E_0}{4L} \int_{0}^{\infty} \int_{0}^{\infty} \hat{E}_1(k,t) \hat{E}_1^*(k,t) \frac{\delta(k-k')}{2\pi} dz e^{i(k-k')z}
\]
\[
= \frac{E_0}{2L} \int_{0}^{\infty} dk \hat{E}_1(k,t) \hat{E}_1^*(k,t)
\]

3. Thus
\[
\langle W_E \rangle = \int_{0}^{\infty} dk \mathcal{E}(k,t)
\]
where we define the Spectral Energy Density
\[
\mathcal{E}(k,t) = \frac{E_0}{2L} |\hat{E}(k,t)|^2
\]

4. We may write the time dependence of \(\mathcal{E}(k,t)\) as
\[
\mathcal{E}(k,t) = \frac{E_0}{2L} |\hat{E}(k,t)|^2 e^{2i\gamma(k,t)t}
\]

b. This implies
\[
\frac{d\mathcal{E}(k,t)}{dt} = 2i\gamma(k,t) \mathcal{E}(k,t)
\]
\(\gamma(k,t)\) is treated as a constant

C. Putting it all together:

1. We have
\[
\frac{\partial \langle P_s \rangle}{\partial t} = \frac{E_0}{\hbar m^2 c^2} \int_{0}^{\infty} dk \frac{i\frac{\partial}{\partial t} \mathcal{E}(k,t)}{\omega = kv^2} \mathcal{E}(k,t)
\]
where we note \(\hat{E}_1(k) \hat{E}_1^*(k) = |\hat{E}(k)|^2\)
Lecture #20 (Continued)

III C. (Continued)

2. Finally we obtain

\[
\frac{\partial \langle f_s \rangle}{\partial t} = \frac{2}{\partial V_z} \left[ D_q(V_z, t) \frac{\partial \langle f_s \rangle}{\partial V_z} \right]
\]

where the Quasilinear Diffusion Coefficient is

\[
D_q(V_z, t) = \frac{2}{\varepsilon_0} \left( \frac{q_s}{m_s} \right)^2 \int_{-\infty}^{\infty} \frac{i \varepsilon(k, t)}{\omega - kV_z} dk
\]

D. Reality of Quasilinear Diffusion Coefficient

1. NOTE: \( \frac{i \varepsilon(k, t)}{\omega - kV_z} = \frac{i \varepsilon(\omega_i - kV_z, -i\gamma)}{(\omega_i + i\gamma - kV_z)(\omega_i - i\gamma - kV_z)} \)

\[
= \frac{i \varepsilon(\omega_i - kV_z)}{(\omega_i - kV_z)^2 + \gamma^2}
\]

2. \[\int_{-\infty}^{\infty} \frac{i \varepsilon(k, t)}{(\omega v(k, t) - kV_z)^2 + \gamma^2} dk\]

\[
= \int_{-\infty}^{\infty} \frac{i \varepsilon(k, t)}{(\omega v(k, t) - kV_z)^2 + \gamma^2} dk
\]

a. Since first term is overall odd, it cancels in integration over \( \int_{-\infty}^{\infty} \) dk.

b. Thus, we are left with a real quantity:

\[
D_q(k, t) = \frac{2}{\varepsilon_0} \left( \frac{q_s}{m_s} \right)^2 \int_{-\infty}^{\infty} \frac{\varepsilon(k, t) \gamma(k, t)}{(\omega v(k, t) - kV_z)^2 + \gamma^2} dk
\]
III. Application of Quasilinear Theory:

A. Initial System \( f_0(v_z) \)

2. Three Quantities must be evolved:
   a. \( \langle f_s(v_z,t) \rangle \) Mean Distribution
   b. \( E(k,t) \) Special Energy Density
   c. \( \gamma(k,t) \) Growth Rate

3. Evolution Equations:
   a. \[ \frac{\partial \langle f_s \rangle}{\partial t} = \frac{2}{\partial v_z} \left[ D_2(v_z, t) \frac{\partial \langle f_s \rangle}{\partial v_z} \right] \]

\[ D_2(v_z, t) = \frac{2}{\mathcal{E}_0} \left( \frac{a_s}{m_s} \right)^2 \int_{-\infty}^{\infty} dk \left( \frac{E(k,t)}{(\omega(k,t) - kv_z)^2 + \omega(k,t)} \right) \]

b. \[ \frac{\partial E(k,t)}{\partial t} = 2 \gamma(k,t) E(k,t) \]

c. \[ D(k,a) = 1 - \frac{\mathcal{E}_0 a_s^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial \langle f_s \rangle}{\partial v_z} \frac{1}{v_z - \omega(k,t)} = 0 \]

The solution of the dispersion relation yields \( \gamma(k,t) \) for unstable modes.

Thus, \( \langle f_s(v_z,t) \rangle, E(k,t) \), and \( \gamma(k,t) \) must be advanced in time self-consistently.
B. The Bump-on-Tail Instability

1. Initial Distribution

2. Assume \( |\chi| \ll (\pi r) \)
   \( \Rightarrow \) Weak Growth Rate:

3. Remember from page 14 II.C.5., in the weak growth approximation,
   \[
   \gamma = \pi \frac{k}{(k^2 + \omega_r^2)^{1/2}} \leq \frac{\omega_r^2}{k^2} \frac{\partial f_{so}}{\partial V_2} \bigg|_{V_2 = \frac{\omega_r}{k}}
   \]

b. Thus \( \gamma > 0 \) only where \( \frac{\partial f_{so}}{\partial V_2} > 0 \) (for \( V_2 > 0 \))

4. Since \( \frac{\partial \Sigma(k, \lambda, \tau)}{\partial \tau} = 2 \Lambda(k, \lambda) \Lambda(k, \lambda) \), the spectral energy density \( \Lambda(k) \) only grows in this range of unstable waves with \( V_2 = \frac{\omega_r}{k} \).

5. For a small, unstable growth rate \( |\chi| \ll (\pi r) \), we can estimate
   \[
   \frac{\gamma}{(\omega_r - kV_2)^2 + \delta^2} \approx \pi \delta(\omega_r - kV_2)
   \]

We can thus evaluate \( D_\gamma(V_2, t, \tau) \)

\[
D_\gamma(V_2, t, \tau) = \frac{2}{E_0 (\pi s)^2} \int_{-\infty}^{\infty} \Lambda(k, t, \tau) \delta(\omega_r - kV_2) \]
6. Therefore
\[ \frac{\partial \langle f_{s} \rangle}{\partial t} (v_{2}, \tau) = \frac{2}{\tau} \left\{ \frac{2 \pi (\tau_{s})^{2}}{E_0} \right\} \sum_{k = \frac{\omega_{0}}{v_{2}}} \left( k = \frac{\omega_{0}}{v_{2}} \right) \frac{\partial \langle f_{s} \rangle}{\partial v_{2}} \]

For each \( v_{2} \), only those solutions of dispersion relation with \( v_{2} = \frac{\omega}{k} \) lead to diffusion.

b. The equation is of the form:
\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial v_{2}} \left[ \frac{v_{2}}{\kappa} \frac{\partial f}{\partial v_{2}} \right] \sim \kappa \frac{\partial^{2} f}{\partial v_{2}^{2}} \]

This, diffusion in velocity space!

This diffusion serves to smooth out the distribution function in velocity space in regions where unstable waves exist, i.e., \( \frac{\partial f}{\partial v_{2}} > 0 \).

7. The Result

a. Since particles must be conserved, area under curve must not change.

b. Since diffusion occurs only in regions where \( \frac{\partial \langle f_{s} \rangle}{\partial v_{2}} > 0 \), the effect of quasilinear diffusion is to evolve \( \langle f_{s} \rangle \) to a state where \( \frac{\partial \langle f_{s} \rangle}{\partial v_{2}} = 0 \) everywhere.

c. At this point, you reach marginal stability. Unstable waves are no longer generated, and quasilinear diffusion stops.