I. Introduction

A. Turbulence arises in many plasma environments:
   a. Fusion plasmas in the laboratory
   b. Solar Wind and Solar Corona
   c. The Interstellar Medium — Star Forming Molecular Clouds
   d. Solar Interior — Solar Convection Zone
   e. Accretion Disks — MRI
   f. Galaxy Clusters

2. Turbulence plays an important role in plasmas because it governs the transport of:
   a. Mass (loss of confinement, mixing, accretion)
   b. Momentum (accretion disks, collisionless shocks, astrophysical jets)
   c. Energy (energy flow, plasma heating)

3. Example: Black Hole Accretion Disks
   a. Magneto-Rotational Instability (MRI) taps free energy in Keplerian differential rotation to drive turbulence
   b. Angular momentum is transported outward, allowing mass to fall inward and accrete
   c. Energy released as matter falls into the deep gravitational potential powers the turbulence
   d. A nonlinear cascade of turbulent energy transfers the energy of fluctuations to ever smaller length scales
   e. At characteristic kinetic length scales, wave-particle interactions (Landau damping, cyclotron damping) transfer energy or electromagnetic fluctuations to structure in distribution function
   f. Ultimately, collisions smooth out the bumps in the distribution function, leading to irreversible thermodynamic plasma heating
9. The hot plasma radiates away some of its heat. This is what we observe from Earth.

4. We will focus today on understanding the turbulent cascade of energy from the large, driving scales to the small, dissipative scales.

DEF: Inertial range: The range of scales in turbulence unaffected by driving or dissipation (bounded by the driving and dissipation scales). In this range, the physics is assumed to be self-similar.

II. Hydrodynamic Turbulence 

Kolmogorov, 1941

A. Example: Mixing Cream into Coffee

1. Coffee

Unmixed

Stir

mixed

2. With just a couple of short stirrings around the cup, the cream rapidly mixes in smoothly.

B. If you do the same thing with paint:

Unmixed

Stir (a couple of times)

Not well mixed.
3. The difference is due to turbulence!  
   \[ \text{Flow in coffee is turbulent, flow in paint is laminar.} \]

4. So, what does the turbulence do?

B. Phenomenological Picture of Turbulence

1. Turbulent motion is stirred at large, driving scale.
   Nonlinear interactions lead to transfer of energy to smaller scales.

2. Cascade continues to ever smaller scales until some dissipative mechanism can damp the turbulence.
   a. In a hydrodynamic fluid (water), viscosity damps the turbulent motions.

C. Kolmogorov's Model of Turbulence (1941)

1. Kolmogorov Hypothesis
   a. Energy transfer is local.
   b. Energy cascade rate through inertial range is constant.
2. Local energy transfer:
   a. Transfer is dominated by interactions between similar scales, i.e., from scale \( l \to \frac{l}{2} \) not \( l \to \frac{l}{16} \)

3. Concept Cascade Rate: In steady state, if cascade rate were not constant (as a function of scale), then energy would either build up or diminish at a certain scale \( \Rightarrow \) this would not be a steady state.

4. DEF: Eddy Turn-around time \( \tau \sim \frac{\varepsilon}{V} \)

   \[ \frac{l}{v} \]

   \begin{align*}
   \text{NOTE: In our scaling arguments here, we drop all factor of order unity:} \\
   \tau = \frac{\pi l}{v} \sim \frac{l}{V}
   \end{align*}

5. Assume energy at a given scale \( l \) is transferred to a smaller scale \( \frac{l}{2} \) over the eddy turn-around time \( \tau \).
   a. Energy Cascade rate \( \varepsilon \sim \frac{\text{energy}}{\text{time}} \sim \frac{V^2}{\tau} \sim \frac{V^3}{(\frac{\pi l}{v})} \)
   b. NOTE: Assume incompressible fluid, so \( \rho = \rho_0 \).
      Thus, kinetic energy (per unit volume) \( E = \frac{1}{2} \rho_0 V^2 \sim V^2 \)
      Since the mass density will always be the same
      c. Since \( E = \text{constant} = E_0 \), then \( E_0 \sim \frac{V^3}{l} \Rightarrow V \sim E_0^{\frac{1}{3}} \frac{V^3}{l} \)
         \[ \Rightarrow V \sim E_0^{\frac{1}{3}} l^{\frac{1}{3}} \]
         So, we find \( V \sim l^{\frac{1}{3}} \) to achieve constant energy cascade rate.
6. **NOTATION**: All quantities depend on the scale, so we'll denote this using a subscript: \( V_e = V(e) \).

Thus \( V_e = E_0 \sqrt{e} \).

7. **Energy Spectrum (1-D)**:
   a. **DEF**: 1-D Energy Spectrum: \( E(k) \) such that \( E = \int E(k) \, dk \).

   Thus, \( E(k) \) has units \( \left[ \frac{E}{k} \right] \).

   b. Theories often use a 1-D Wavenumber Spectrum & Kinetic Energy to study turbulence:

   ![Graph showing the log-log plot of energy spectrum with labeled axes: \( \log[E_k] \) vs. \( \log[k] \).]

   - **DEFSpectral Index**: Slope of spectrum on log-log plot.
   - **Dissipation Scale**
   - **Inertial Range**: \( \frac{1}{L} \)
   - **Driving Scale**: \( e \)
   - **Viscous Scale**: \( e \nu \)

   c. Self-similar physics yields power law behavior

   \( \Rightarrow \) Straight-line on log-log plot of \( \log E(k) \) vs. \( k \).

   d. We want to find the spectral index

8. **Kolmogorov Spectrum**
   a. \( E_k \propto \frac{V^2}{k} \Rightarrow E_k \propto \frac{E_0^3 k^{\frac{5}{3}}}{k} \Rightarrow E_k \sim E_0^{\frac{2}{3}} k^{-\frac{5}{3}} \) \( \Rightarrow \) Kolmogorov Spectrum

   b. Note: \( k = \frac{2\pi}{L} \sim \frac{1}{L} \), so \( V_e^2 \propto E_0 \sqrt{e} \) \( \Rightarrow \) \( V_e \propto E_0 \sqrt{e} \) Kolmogorov Index.
Lecture 24 (Continued)

II. (Continued)

D. Summary: 1.

C. NOTE: \( T_k \sim \frac{1}{kv_k} \sim \frac{1}{k(c_s^3 k^3)} \sim \frac{1}{k^{3.3}} \)

\[ \Rightarrow \text{Cascade time decreases } T_k \propto k^{-3.3} \text{ as energy goes to smaller scale (higher } k) \]

\[ \Rightarrow \text{This is why it only takes a couple of stirring periods at the large scale for the energy to reach the viscous scale (why your ocean mixes into your coffee quickly).} \]

III. MHD Turbulence:

1. Here we are concerned with low-frequency turbulence (\( \omega \ll \omega_A \)), so MHD provides an appropriate description.
2. Presence of a magnetic field established a preferred direction in the plasma \( \Rightarrow \) Anisotropy
   a. We may expect possible anisotropy to arise

   \[ k \quad \Rightarrow \quad k_{\perp} \quad \Rightarrow \quad k_{\parallel} \quad \Rightarrow \quad B_0 \]

   b. We refer to components \( L \parallel \) to \( B_0 \).

B. Alfvén waves play a crucial role in MHD Turbulence.
   a. Rather than a superposition of eddies as in HD Turbulence, we have Alfvén waves travelling up and down \( B_0 \).
C. Two Timescales in MHD Turbulence

1. Nonlinear Transfer Rate: Analogous to eddy turnover time in HD turbulence. \( \omega_e \sim \frac{1}{T_e} \sim k_1 \lambda_1 \)

2. Linear Alfvén Wave Frequency:

\[ \omega = k_1 \lambda_1 \]

3. We need to understand how these two timescales interplay in MHD turbulence.

D. Key Concepts in MHD Turbulence

In addition to the Kolmogorov Hypothesis, there are three new key concepts that must be introduced to study MHD turbulence:

1. Kraichnan Hypothesis: The large-scale magnetic field behaves like a mean field for small-scale fluctuations.

Any turbulent MHD system may be viewed, on the small scales, as a collection of Alfvén waves on a mean field.

2. Anisotropic Cascades: Energy transfer occurs preferentially to small length scales perpendicular to the mean field \( B_0 \).

This means when \( \frac{1}{\kappa_1} \approx \lambda_1 \)\( \Rightarrow \) Dominated by \( k_1 \)
3. Critical Balance: The linear ($\omega = k V_A$) and nonlinear ($\omega \sim k V_A$) timescales remain in a state of critical balance as the turbulence cascades to smaller scale. (Goldreich & Sridhar, 1995)

F. Alfvén Wave Packet Collisions

The fundamental building blocks of MHD Turbulence are collisions between oppositely directed Alfvén Wave packets.

1. In incompressible MHD turbulence, the nonlinear interactions occur only between oppositely directed wave packets (Kraichnan, 1965)

![Diagrams of wave packets passing through each other and nonlinear interactions leading to energy transfer to higher k (smaller scale)]

F. Weak Turbulence:

1. Before: Weak

2. In weak turbulence, it takes many collisions to transfer energy to smaller scale $\Rightarrow$ slower cascade rate.
III. Strong MHD Turbulence:

1. All energy transfers to small scale during a single collision.
   a. NOTE: collision time $T_c \sim \frac{1}{\omega_c} \sim \frac{1}{K_1 V_A}$
   $\Rightarrow T_c \sim T_{he} \Rightarrow c_1 \sim c_{he}$
   b. Nonlinear transfer time $T_{nc} \sim \frac{1}{c_{nc}} \sim \frac{1}{K_1 V_1}$
      $\Rightarrow$ Critical Balance

2. Estimate nonlinear energy transfer rate:
   a. $E = \frac{\text{energy}}{\text{time}} \sim \frac{V_1^2}{(K_1 V_1)} \sim K_1 V_1^2 = \text{const} = E_0$
   b. $V_1 \sim E_0^{\frac{1}{3}} K_1^{\frac{1}{3}}$

3. Anisotropic 1-D Wavenumber Spectrum: $E = \int_{k_{lo}}^{k_{ho}} E_k(k) d(k)$

   ![Graph showing $E_k \propto k_1^{-5/3}$ vs $\log(k_1)$]

   $$E_k \propto \frac{E_0^2}{k_1} \frac{V_1^2}{K_1^2} = \frac{E_0^2}{K_1^2} \frac{V_1^2}{k_1^2}$$

   $$E_k \sim E_0^{\frac{2}{3}} K_1^{\frac{5}{3}}$$

   Godreich-Sridhar Spectrum

4. What about spectrum in $k_1$?
   a. Use critical balance: $c_1 \sim c_{he} \Rightarrow k_1 V_A \sim k_1 V_1 \sim k_1 (E_0^{\frac{1}{3}} K_1^{\frac{1}{3}})$
      $\Rightarrow k_1 \sim \frac{E_0^{\frac{1}{3}} K_1^{\frac{1}{3}}}{V_A}$
   b. If at driving scale $k_1 = k_1 = k_0$, then $E_0 = k_0 V_A^3$ since $k_0 V_A = k_0 V_1$
   a. Thus $k_1 \sim \left( \frac{k_0 V_A}{V_A} \right)^{\frac{1}{3}} \Rightarrow k_1 \sim k_0^{\frac{1}{3}} k_1^{\frac{2}{3}}$
5. Anisotropic Cascade in Wavevector Space \((k_1, k_{11})\)

a. **NOTE**: Physics is anisotropic about mean field \(B_0\), so
2-D wavevector \(k\)-space collapses to 2-D \((k_1, k_{11})\).

\[
\begin{align*}
\log(k_1) & \quad \log(k_{11}) \\
0 & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7
\end{align*}
\]

At driving scale,
\[k_{11} \approx k_1 \approx 1\]

Driving Scale

\[\text{Energy} \quad \text{Cascade of} \quad \text{Energy} \]

\[\log(k_{11}) \quad \log(k_1)\]

Due to considerations of weak turbulence, fluctuations fill in all space with \(k_{11} \leq k_1^{2/3} k_{1/3}\)

\[\begin{align*}
\omega \approx \text{same} \\
\omega \ll \text{same}
\end{align*}\]

GS cone of turbulence (Anisotropic).

6. Implications of Anisotropic Cascade:

a. Because the frequency of Alfvénic fluctuations \(\omega = k_{11} V_A\),

\[k_{11} \approx k_1^{2/3}\]

since \(k_{11} \ll k_1^{2/3}\), turbulence fluctuations tend to end at:

1. Highly elongated along magnetic field
   \[l_1 \approx l_4 \Rightarrow k_{11} \ll k_4\]

2. Low-frequency, even as \(l_1 \rightarrow l_{11}\),
   \[\omega \ll \text{omega} \text{ and so cyclotron damping is weak} \Rightarrow \text{Landau damping responsible for dissipation} \text{ (collisionless)}\]