# Lecture #24 Turbulence



## I. Introduction

A. 1. Turbulence arises in many plasma environments:

a. Fusion plasmas in the laboratory

b. Solar Wind and Solar Corona

c. The Interstellar Medium — Star Forming Molecular Clouds

d. Solar Interior — Solar Convection Zone

e. Accretion Disks — MRI

f. Galaxy Clusters

2. Turbulence plays an important role in plasmas because it governs the transport of:

a. Mass (Loss of confinement, mixing, accretion)

b. Momentum (accretion disks, collisionless shocks, astrophysical jets)

c. Energy (energy flow, plasma heating)

3. Example: Black Hole Accretion Disks

a. Magneto-Rotational Instability (MRI) taps free energy in Keplerian differential rotation to drive turbulence

b. Angular momentum is transported outward, allowing mass to fall inward and accrete

c. Energy released as matter falls into the deep gravitational potential powers the turbulence.

d. A nonlinear cascade of turbulent energy transfers the energy of fluctuations to ever smaller length scales

e. At characteristic kinetic length scales, wave-particle interactions (Landau damping, cyclotron damping) transfer energy of electromagnetic fluctuations to structure in distribution functions.

f. Ultimately, collisions smooth out the bumps in the distribution function, leading to irreversible thermodynamic plasma heating.

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I. A.3.C (continued)

g. The hot plasma radiates away some of its heat. This is what we observe from earth.

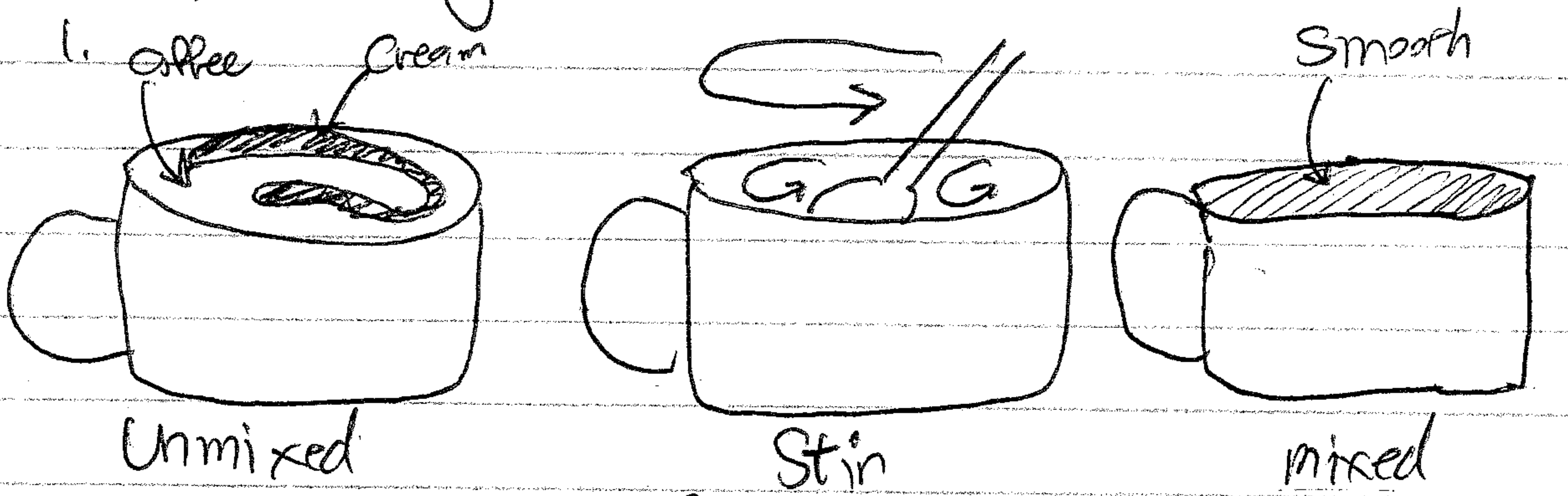
4. We will focus today on understanding the turbulent cascade of energy from the large, driving scales to the small, dissipative scales.

⇒ <u>DEF</u>: <u>Inertial range</u>: The range of scales in turbulence unaffected by driving or dissipation (bounded by the driving and dissipation scales). In this range, the physics is assumed to be <u>self-similar</u>.

## II. Hydrodynamic Turbulence Kolmogorov, 1941

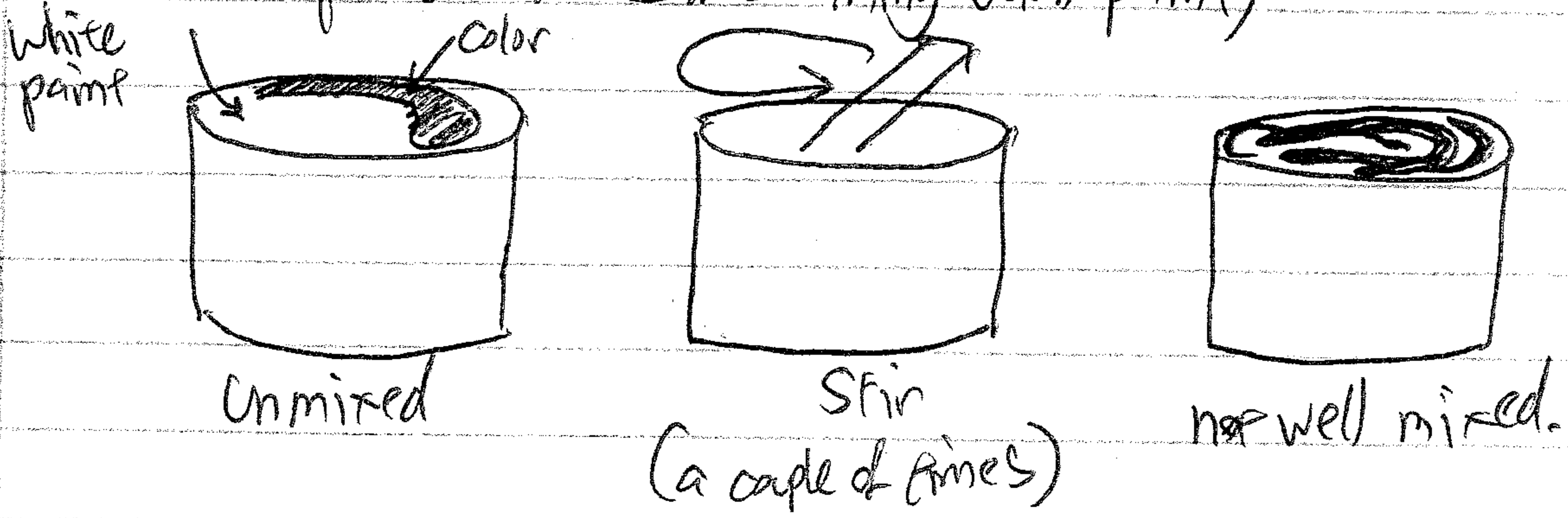
A. <u>Example</u>: Mixing Cream into Coffee

1.




a. With just a couple of short stirrings around the cup, the cream rapidly mixes in smoothly

b. If you do the same thing with paint,



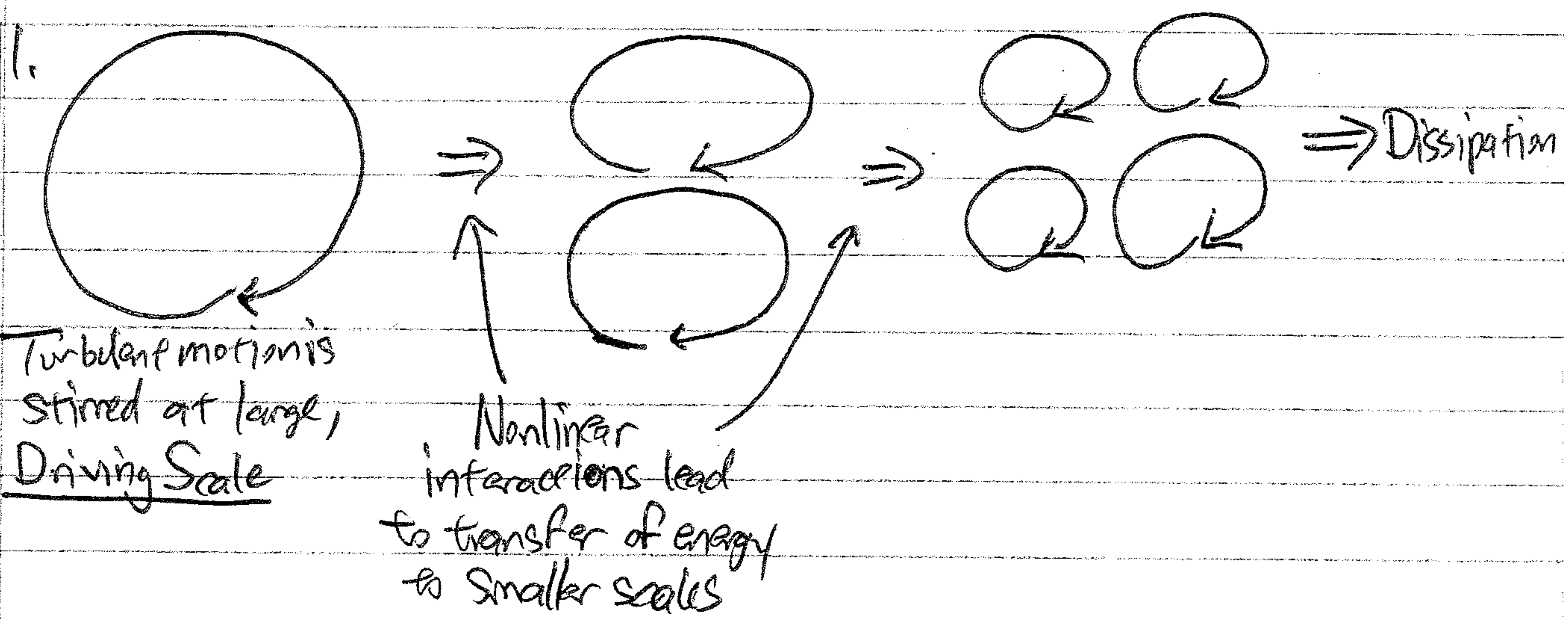
II A (Continued)

3. The difference is due to turbulence!
   ⇒ flow in coffee is turbulent, flow in paint is laminar.

4. So, what does the turbulence do?

## B. Phenomenological Picture of Turbulence

1.




2. Cascade continues to ever smaller scales until some dissipative mechanism can damp the turbulence
   a. In a hydrodynamic fluid (water), viscosity damps the turbulent motions.

## C. Kolmogorov's Model of Turbulence (1941)

1. Kolmogorov Hypothesis
   a. Energy transfer is local.
   b. Energy cascade rate through inertial range is constant.

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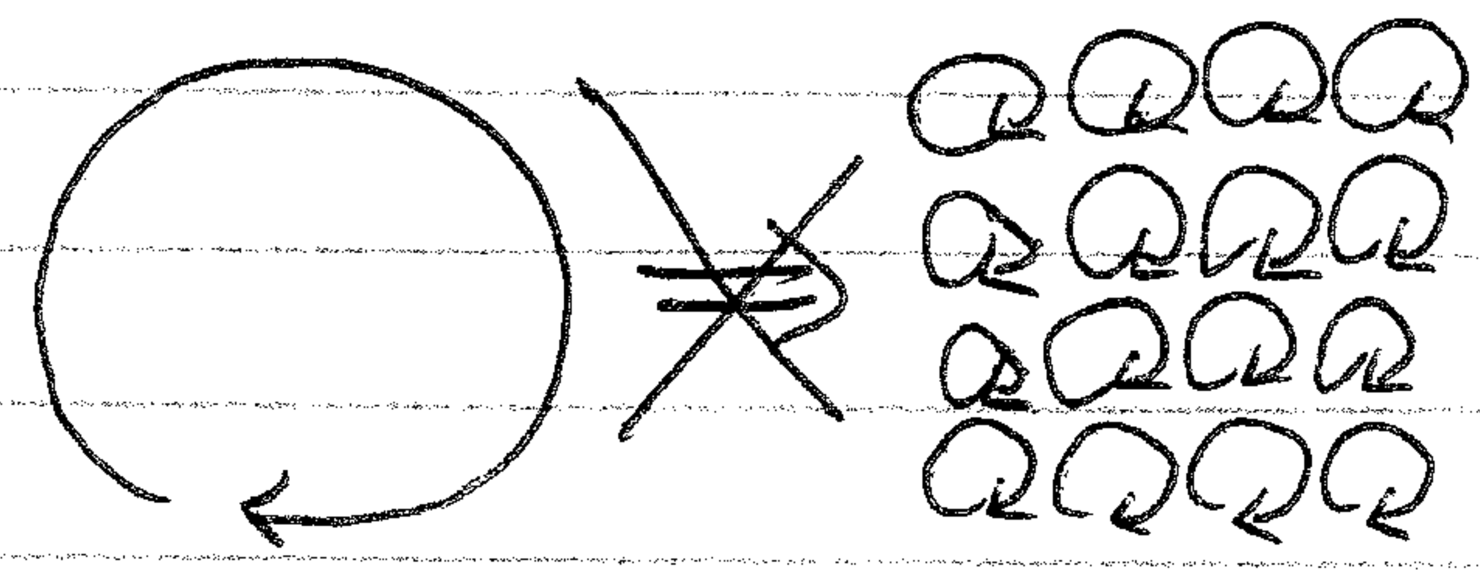
II. C. (Continued)

2. Local energy transfer:



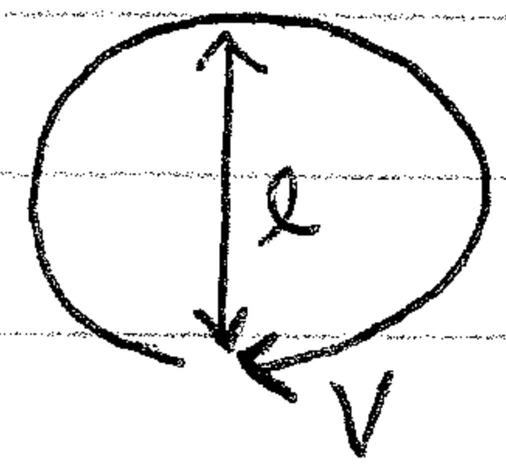
a. Transfer is dominated by interactions between similar scales, ie.

from scale $\ell \rightarrow \ell/2$

not $\ell \rightarrow \ell/16$

3. Constant Cascade Rate: In steady state, if cascade rate were not constant (as a function of scale), then energy would either build up or diminish at a certain scale $\Rightarrow$ this would not be a steady state.

4. DEF: Eddy Turn-around time $\tau \sim \frac{\ell}{v}$



NOTE: In our scaling arguments here, we drop all factor of order unity. $\tau = \frac{\pi \ell}{v} \sim \frac{\ell}{v}$

5. Assume energy at a given scale $\ell$ is transferred to a smaller scale $\ell/2$ over the eddy turnaround time $\tau$.

a. Energy Cascade rate $\epsilon \sim \frac{\text{energy}}{\text{time}} \sim \frac{v^2}{\tau} \sim \frac{v^2}{(\ell/v)} \sim \frac{v^3}{\ell}$

b. NOTE: Assume an incompressible fluid, so $\rho = \rho_0$.

Thus, Kinetic energy (per unit volume) $E = \frac{1}{2}\rho_0 v^2 \sim v^2$

since the mass density will always be the same.

c. Since $\epsilon$ = constant $= \epsilon_0$, then $\epsilon_0 \sim \frac{v^3}{\ell} \Rightarrow \boxed{v \sim \epsilon_0^{1/3} \ell^{1/3}}$

So, we find $v \propto \ell^{1/3}$ to achieve constant energy cascade rate.

II. C (Continued)

6. NOTATION: All quantities depend on the scale, so we'll denote this using a subscript; $v_\ell = v(\ell)$.

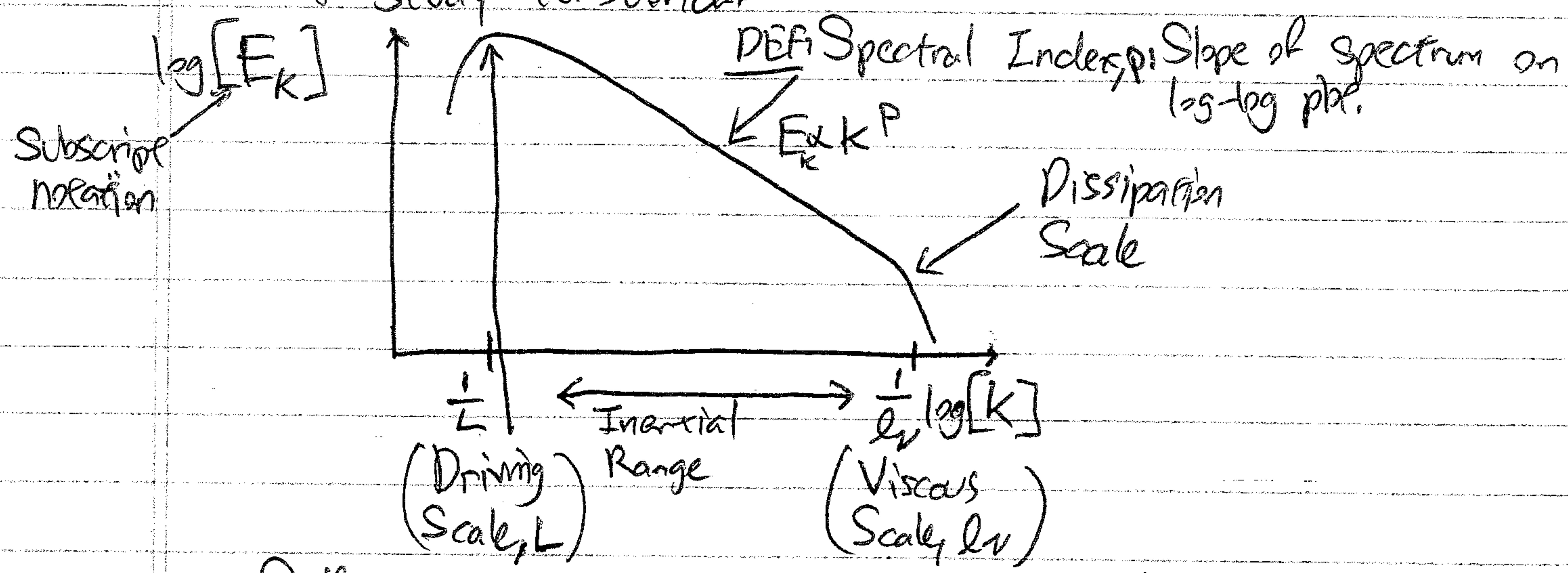
Thus $\boxed{v_\ell = \epsilon_0^{\frac{1}{3}} \ell^{\frac{1}{3}}}$

7. Energy Spectrum (1-D):

a. DEF: 1-D Energy Spectrum: $E(k)$ such that $E = \int_{k_0}^{\infty} E(k)\, dk$

Thus, $E(k)$ has units $\left[\frac{E}{k}\right]$.

b. Theorists often use a 1-D Wavenumber Spectrum of Kinetic Energy to study turbulence:




c. Self-similar physics yields power law behavior

$\Rightarrow$ Straight-line on log-log plot of of $E(k)$ vs. $k$.

d. We want to find the spectral index

8. Kolmogorov Spectrum

a. $E_k \propto \frac{v_k^2}{k} \propto \frac{\epsilon_0^{2/3} k^{-2/3}}{k} \Rightarrow$ $\boxed{E_k \sim \epsilon_0^{2/3} k^{-\frac{5}{3}}}$ Kolmogorov Spectrum

$\Rightarrow -\frac{5}{3}$ is Kolmogorov Index.

b. Note: $k = \frac{2\pi}{\ell} \sim \frac{1}{\ell}$, so $v_\ell = \epsilon_0^{\frac{1}{3}} \ell^{\frac{1}{3}} \Rightarrow v_k = \epsilon_0^{\frac{1}{3}} k^{-\frac{1}{3}}$

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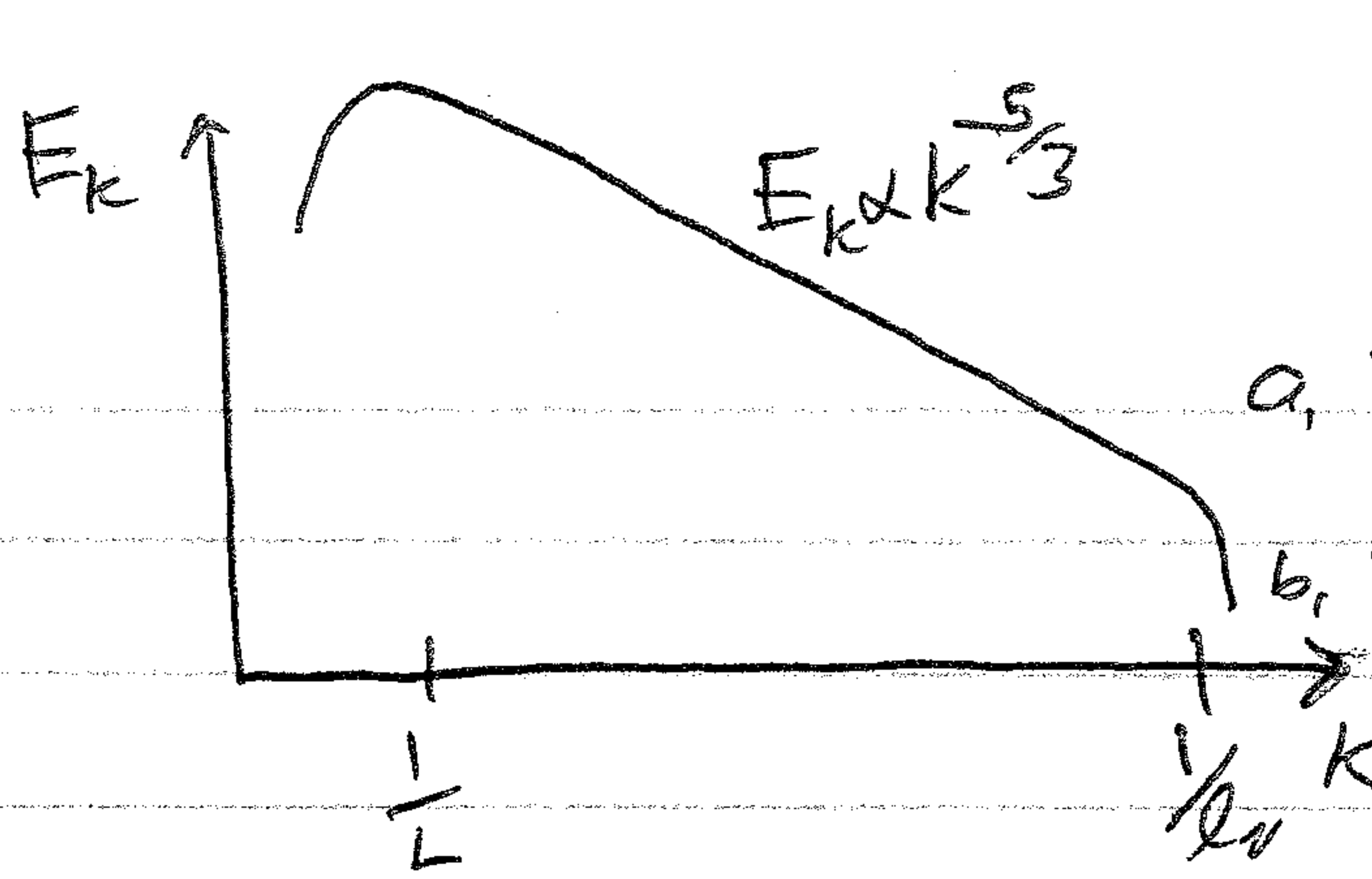
II. (Continued)

D. Summary: 1.




a. $v_k \sim \epsilon_0^{1/3} k^{-1/3}$

b. $\tau_k \sim \frac{1}{k v_k}$

c. NOTE: $\tau_k \sim \frac{1}{k v_k} \sim \frac{1}{k(\epsilon_0^{1/3} k^{-1/3})} \sim \frac{1}{k^{2/3}}$

$\Rightarrow$ Cascade time decreases $\tau_k \propto k^{-2/3}$ as energy goes to smaller scale (higher $k$)

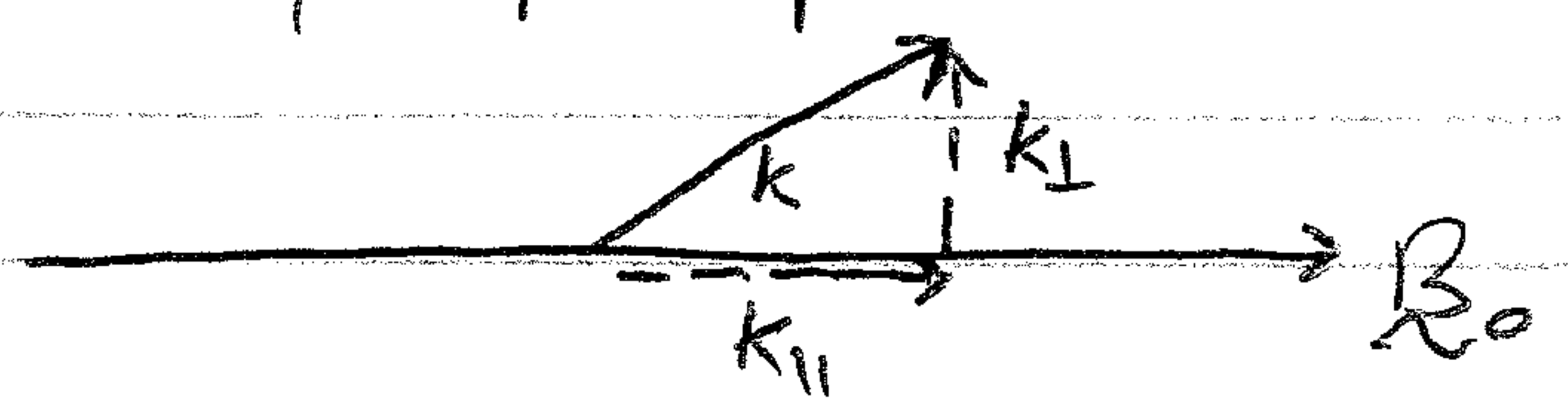
$\Rightarrow$ This is why it only takes a couple of stirring periods at the large scale for the energy to reach the viscous scale (why your cream mixes into your coffee quickly).

III. MHD Turbulence:

A. 1. Here we are concerned with low-frequency turbulence ($\omega \ll \omega_{ci}$), so MHD provides an appropriate description.

2. Presence of a magnetic field established a preferred direction in the plasma $\Rightarrow$ Anisotropy

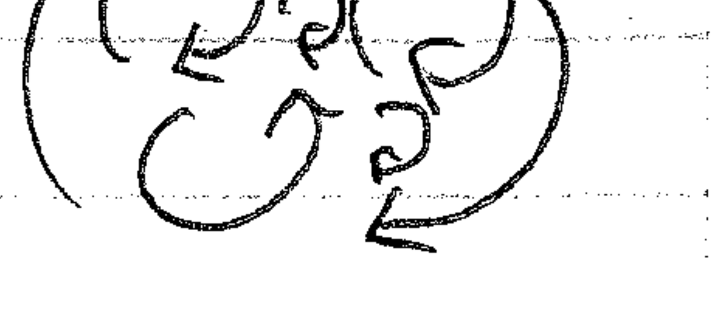
a. We may expect possible anisotropy to arise




b. We refer to components $\perp$ & $\parallel$ to $B_0$

B. Alfven waves play a crucial role in MHD Turbulence.

a. Rather than a superposition of eddies as in HD Turbulence, we have Alfven waves travelling up and down $B_0$.



III. (Continued)

## C. Two Timescales in MHD Turbulence

1. Nonlinear Transfer Rate: Analogous to eddy turnaround time in HD turbulence. $\omega_{nl} \sim \frac{1}{\tau_{nl}} \sim k_\perp v_\perp$
2. Linear Alfven Wave Frequency:

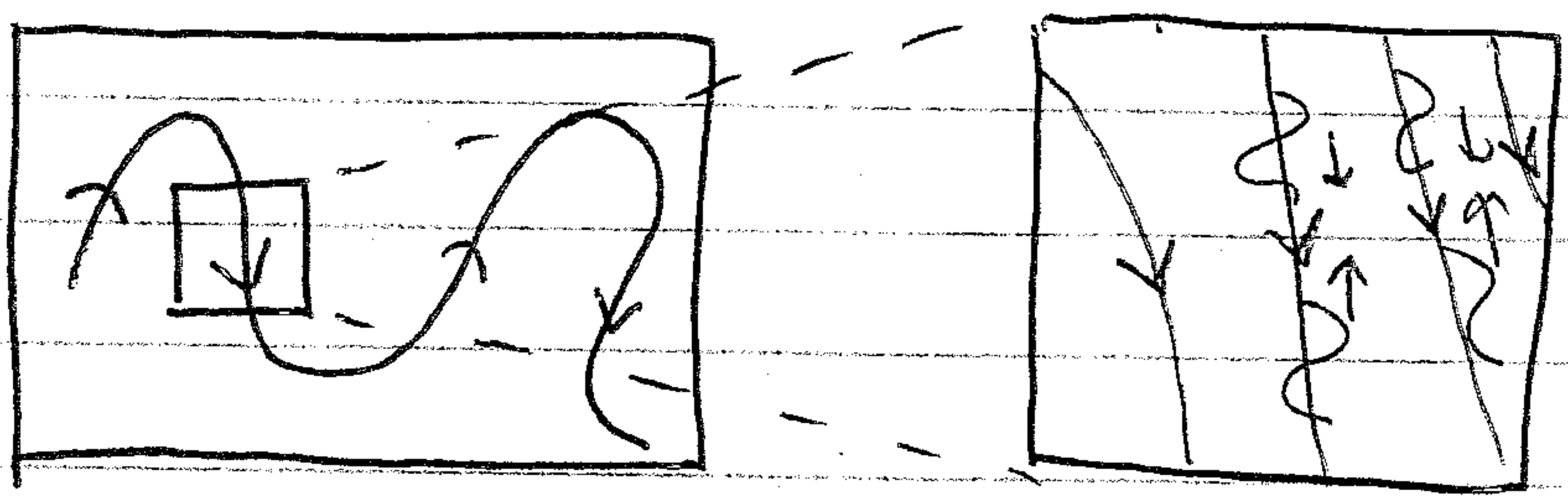
   $\omega = k_\parallel v_A$

3. We need to understand how these two timescales interact in MHD turbulence

## D. Key Concepts in MHD Turbulence

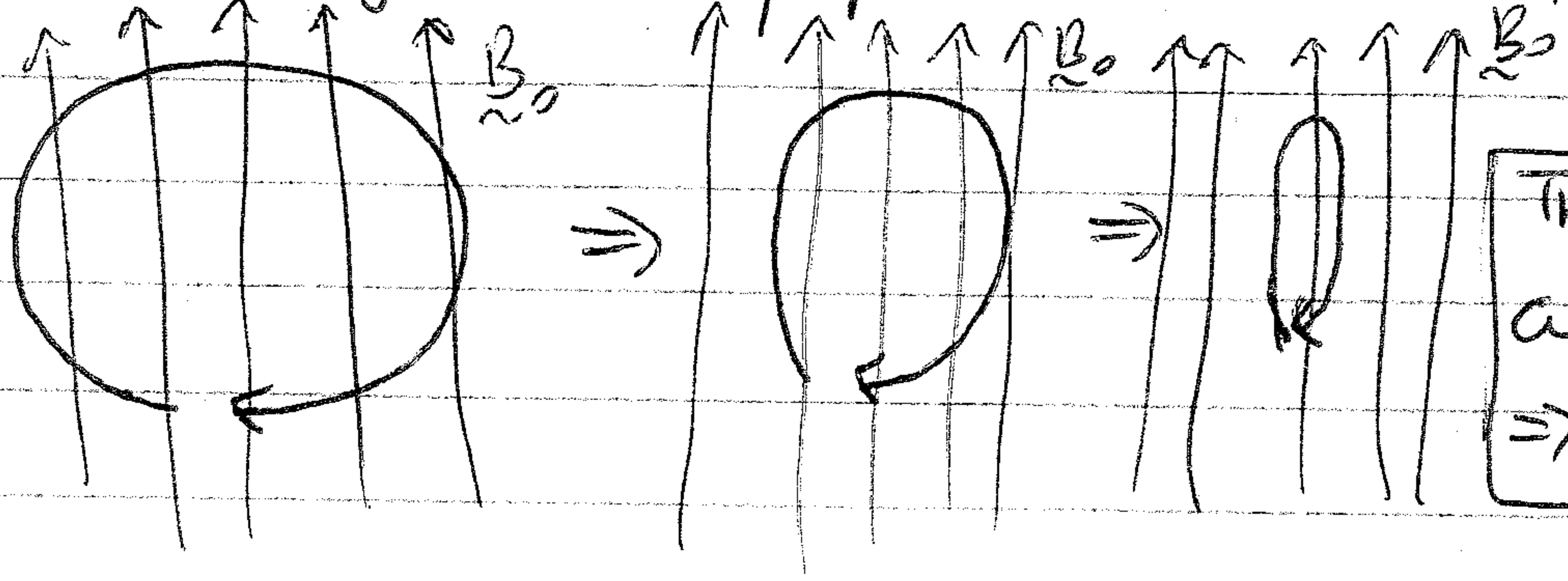
In addition to the Kolmogorov Hypothesis, there are three new key concepts that must be introduced to study MHD turbulence.

1. Kraichnan Hypothesis: The Large-Scale magnetic field behaves like a mean field for small-scale fluctuations:



$\Rightarrow$ Any turbulent MHD system may be viewed, on the small scales, as a collection of Alfven waves on a mean field.

2. Anisotropic Cascade: Energy transfer occurs preferentially to small length scales perpendicular to the mean field $\underset{\sim}{B}_0$




This means $\omega_{nl} \sim \frac{1}{\tau} \sim \cancel{kv} \sim k_\perp v_\perp$

$\Rightarrow$ Dominated by $k_\perp$!

III. D. (Continued)

3. Critical Balance: The linear ($\omega = k_\parallel v_A$) and nonlinear ($\omega_{nl} \sim k_\perp v_\perp$) timescales remain in a state of critical balance as the turbulence cascades to smaller scale.

$$\boxed{\omega \sim \omega_{nl}}$$

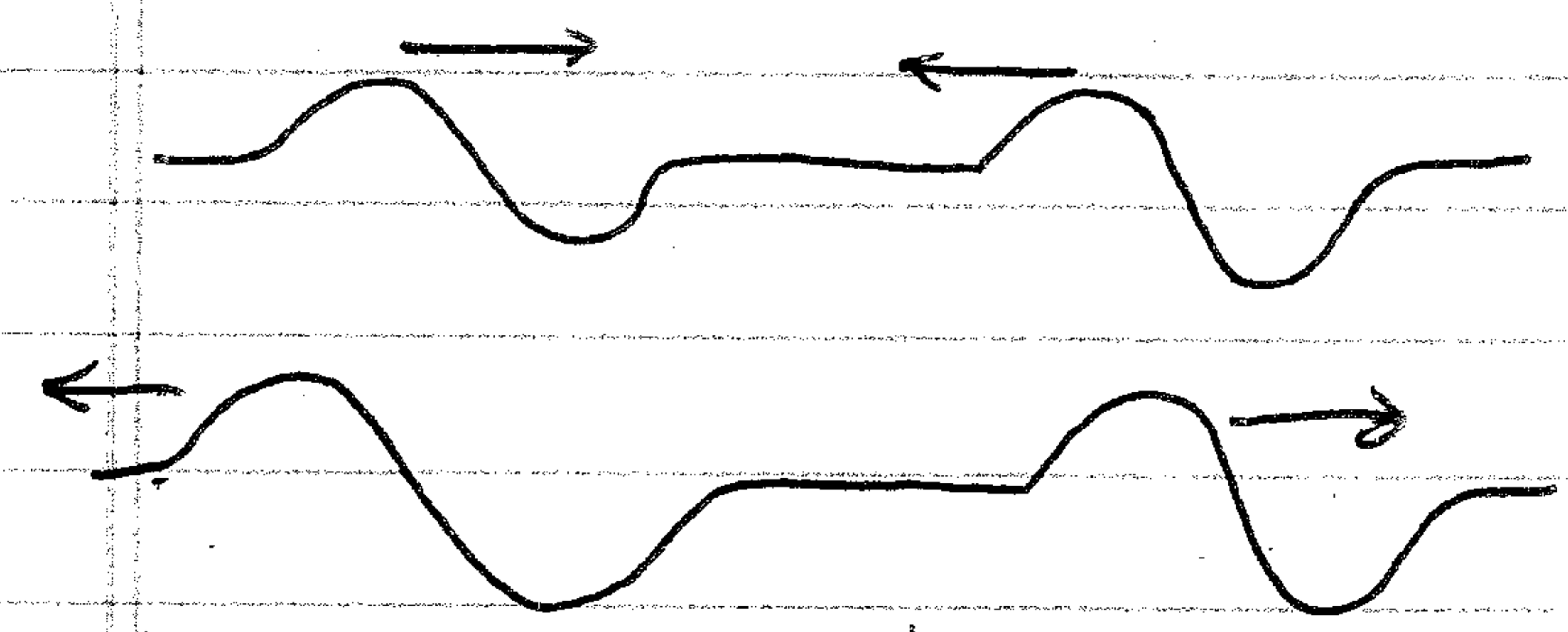
(Goldreich & Sridhar, 1995)

E. Alfven Wave Packet Collisions

The fundamental building blocks of MHD Turbulence are collisions between oppositely directed Alfven Wave packets.

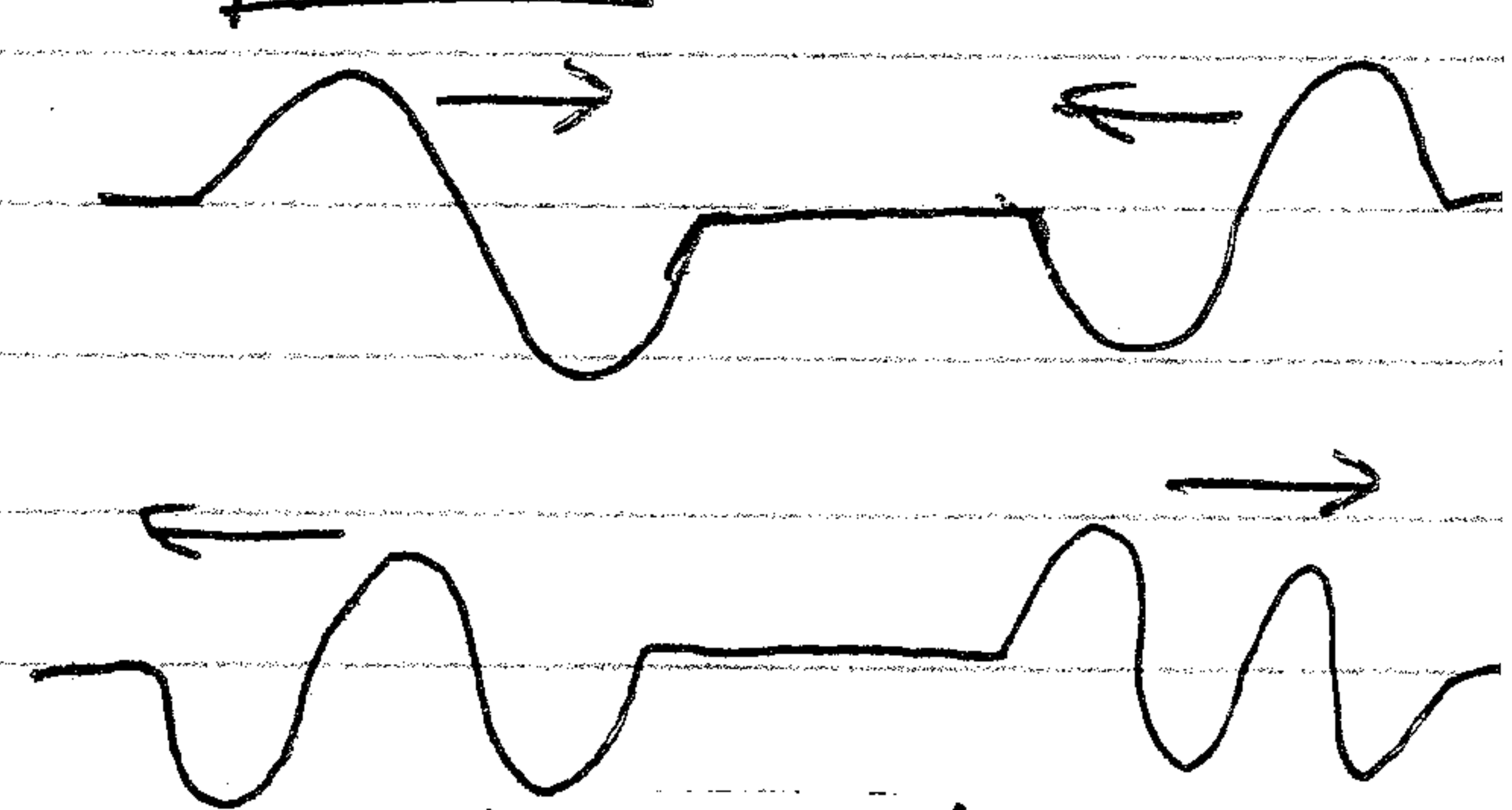
1. In incompressible MHD turbulence, it is an exact result the nonlinear interactions occur only between oppositely directed wave packets (Kraichnan, 1965)

a. Linear



Wave packets pass through each other unchanged.

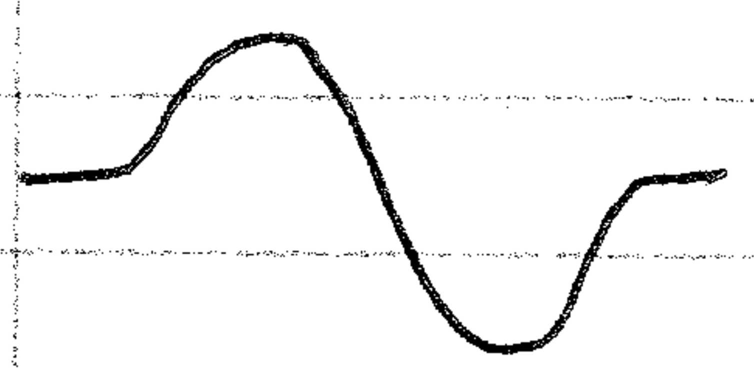
Nonlinear:



Nonlinear interactions have lead to energy transferred to higher $k$ (smaller scale).

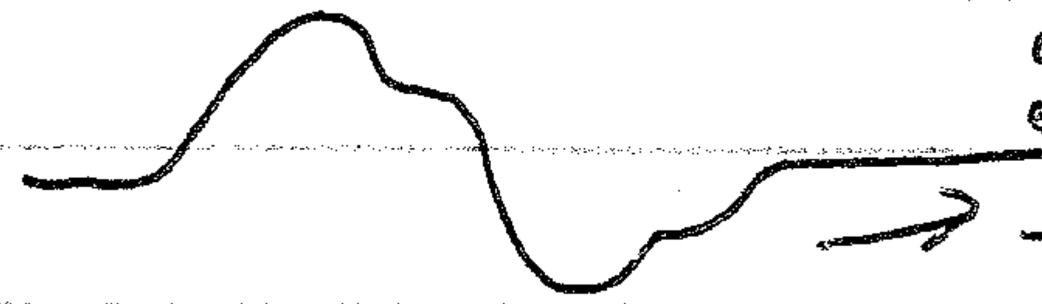
F. Weak Turbulence:

1. Before:

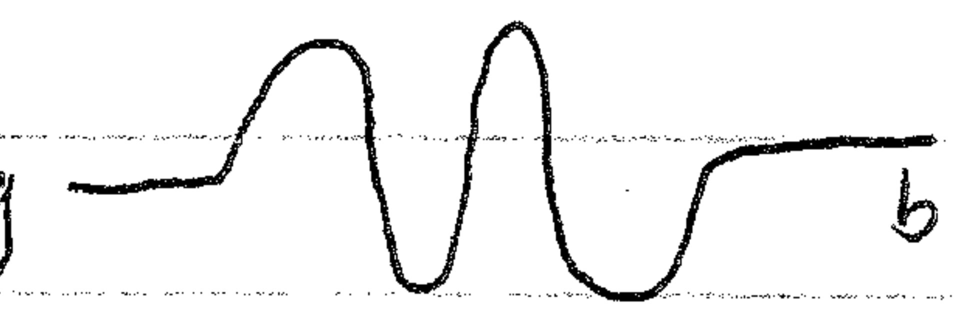


After:

Weak



Strong



Weak

a. only a small fraction of energy has transferred to smaller scale in a single collision

Strong

b. All energy has transferred to smaller scale in a single collision.

2. In weak turbulence, it takes many collisions to transfer energy to smaller scale ⇒ slower cascade rate!

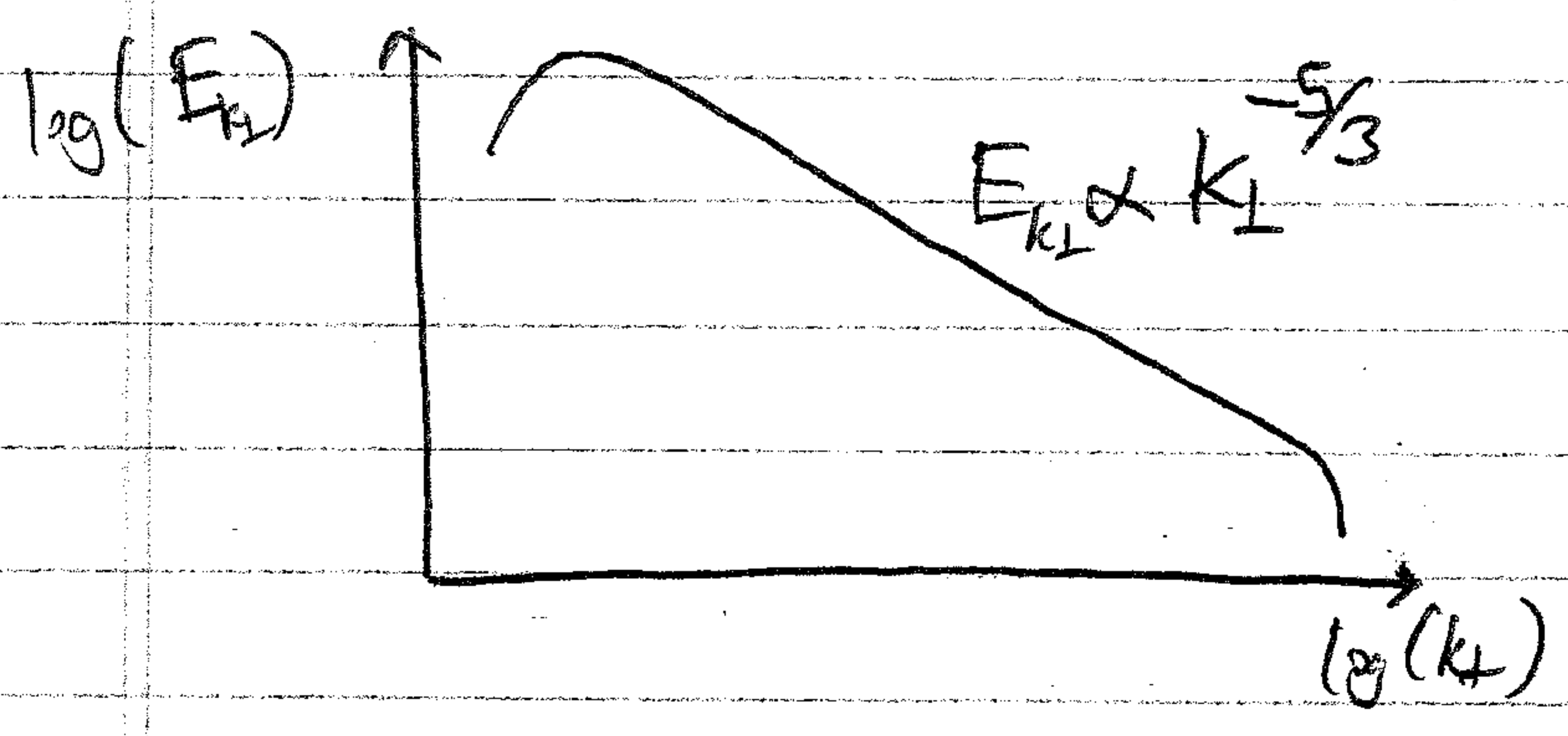
III G. Strong MHD Turbulence:

1. All energy transfers to small scale during a single collision.
   a. NOTE: collision time $\tau_c \sim \frac{1}{\omega} \sim \frac{1}{k_\parallel v_A}$
   b. Nonlinear transfer time $\tau_{nl} \sim \frac{1}{\omega_{nl}} \sim \frac{1}{k_\perp v_\perp}$

   $\left.\right\} \tau_c \sim \tau_{nl} \Rightarrow \omega \sim \omega_{nl}$ $\Rightarrow$ Critical Balance

2. Estimate Nonlinear energy transfer rate:
   a. $\epsilon = \frac{\text{energy}}{\text{time}} \sim \frac{v_\perp^2}{\left(\frac{1}{k_\perp v_\perp}\right)} \sim k_\perp v_\perp^3 = \text{Constant} = \epsilon_0$
   b. $v_\perp \sim \epsilon_0^{\frac{1}{3}} k_\perp^{-\frac{1}{3}}$

3. Anisotropic 1-D Wavenumber Spectrum: $E \equiv \int_{k_{\perp 0}}^{\infty} dk_\perp \, E_{k_\perp}(k_\perp)$



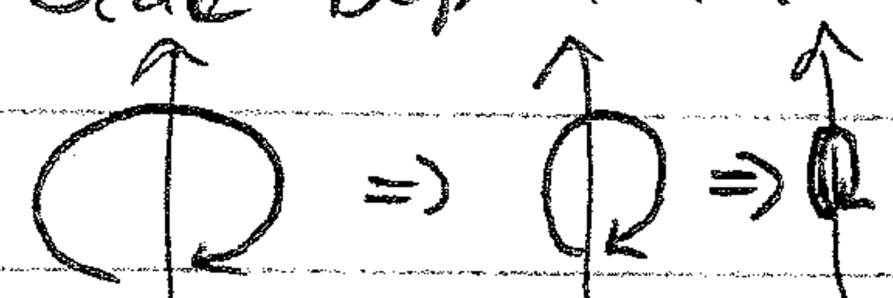

   a. $E_{k_\perp} \propto \frac{v_\perp^2}{k_\perp} \propto \frac{\epsilon_0^{2/3} k_\perp^{-\frac{2}{3}}}{k_\perp} \propto \epsilon_0^{2/3} k_\perp^{-\frac{5}{3}}$

   $$\boxed{E_{k_\perp} \sim \epsilon_0^{2/3} k_\perp^{-5/3}}$$

   Goldreich-Sridhar Spectrum

4. What about spectrum in $k_\parallel$?
   a. Use critical balance: $\omega \sim \omega_{nl} \Rightarrow k_\parallel v_A \sim k_\perp v_\perp \sim k_\perp (\epsilon_0^{\frac{1}{3}} k_\perp^{-\frac{1}{3}})$

   $\Rightarrow k_\parallel \sim \frac{\epsilon_0^{\frac{1}{3}}}{v_A} k_\perp^{2/3} \Rightarrow \boxed{k_\parallel \propto k_\perp^{2/3}}$ Scale Dependent Anisotropy!

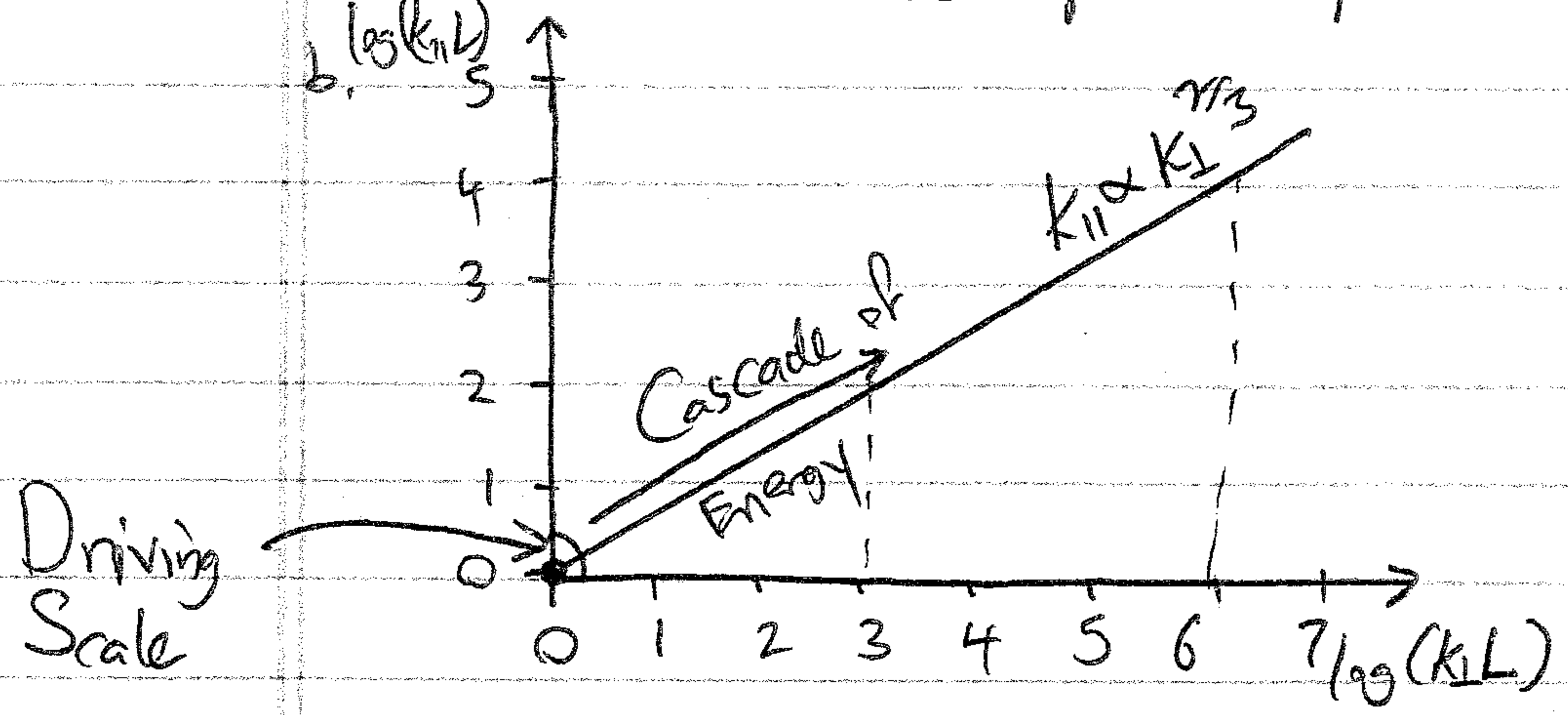


   b. If at driving scale $k_\parallel = k_\perp = k_0$, then $\epsilon_0 = k_0 v_A^3$ since $k_0 v_A = k_0 v_\perp \Rightarrow v_\perp = v_A$
   a. Thus $k_\parallel \sim \frac{(k_0 v_A^3)^{\frac{1}{3}}}{v_A} k_\perp^{2/3} \Rightarrow \boxed{k_\parallel \sim k_0^{\frac{1}{3}} k_\perp^{2/3}}$

III G. (Continued)

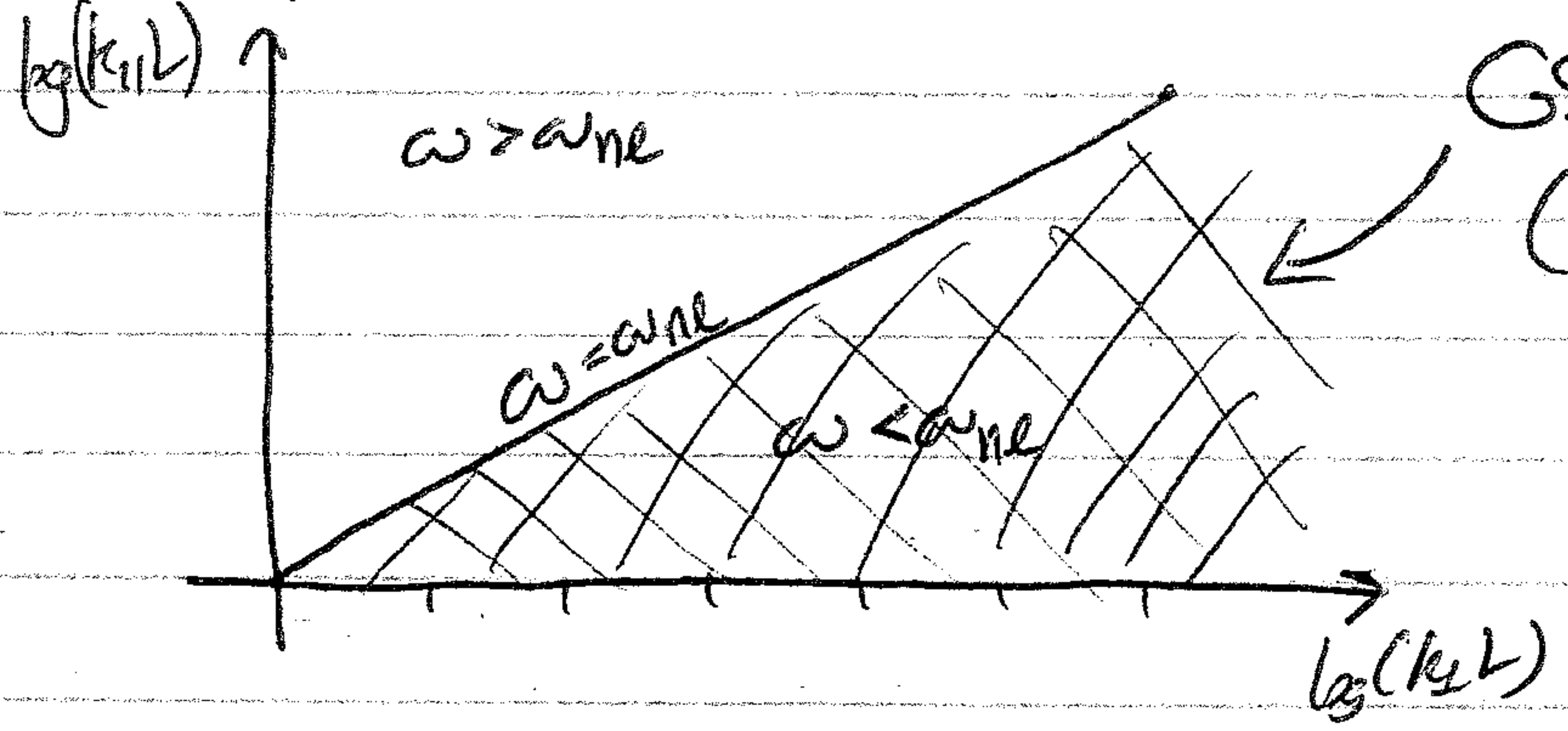
5. Anisotropic Cascade in Wavenumber Space $(k_\perp, k_\parallel)$

a. NOTE: Physics is axisymmetric about mean field $\mathbf{B}_0$, so 3-D wavevector $\mathbf{k}$-space collapses to 2-D $(k_\perp, k_\parallel)$.

b. 


At driving scale, $k_\parallel L = k_\perp L = 1$

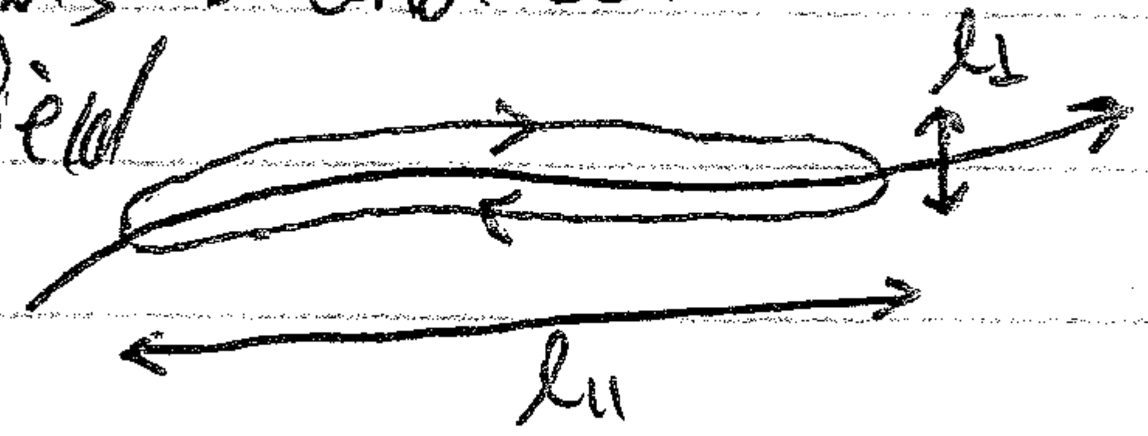
c. Due to considerations of weak turbulence, fluctuations fill in all space with $k_\parallel \leq k_\perp^{2/3} k_0^{1/3}$




6. Implications of Anisotropic Cascade:

a. Because the frequency of Alfvénic fluctuations $\omega = k_\parallel v_A$, since $k_\parallel \propto k_\perp^{2/3}$, turbulence fluctuations tend to be:

1. Highly elongated along magnetic field
$l_\parallel \gg l_\perp \Rightarrow k_\parallel \ll k_\perp$




2. Low-frequency, even as $l_\perp \to r_{Li}$.
This means $\omega \ll \omega_{ci}$ and so cyclotron damping is weak $\Rightarrow$ Landau damping responsible for dissipation (collisionless)