Lecture 25: Weak and Strong MHD Turbulence

I. Review of Kolmogorov's Model for Hydrodynamic Turbulence

A. 1. Phenomenological Picture

\[ \rightarrow \rightarrow \rightarrow \frac{\partial^2}{\partial t^2} \]

2. Kolmogorov's
   a. Energy transfer is local (in sub-space)
   b. Energy cascade rate is constant in inertial range

3. a. Turnaround time: \[ T = \frac{\ell}{V} \quad \Rightarrow \quad V \sim \ell^{\frac{3}{2}} \]
   b. \[ E = \frac{V^2}{T} \sim \frac{\ell^3}{2} \quad \Rightarrow \quad E \sim \ell^{\frac{5}{3}} \]
   c. \[ E_k = \frac{V^2}{k} \propto k^{-\frac{5}{3}} \]

\[ E_k \propto k^{-\frac{5}{3}} \]

4. Driving

\[ \rightarrow \rightarrow \rightarrow \frac{\partial^2}{\partial t^2} \]

\[ \rightarrow \rightarrow \rightarrow \frac{\partial^2}{\partial t^2} \]

II. MHD Turbulence: Ironshnikov-Kraichnan

A. Introduction
   1. Ironshnikov (1963) & Kraichnan (1965) independently extended Kolmogorov's turbulence picture to MHD turbulence.
   2. Kraichnan realized the nonlinear terms occur only when oppositely directed Alfvén wave packets interfere.
   a. This is easily seen when Incompressible MHD (\( \psi = 0 \)) is written in Els"asser Variables

\[ \begin{align*}
  \zeta & = \psi = \frac{8\pi}{\sqrt{\mu_0}} \\
  \end{align*} \]
Lecture #25 (Continued)

II. A. 2. (Continued)

b. \[ \frac{\partial \bar{z}_\pm}{\partial t} + (\mathbf{V}_A \cdot \nabla) \bar{z}_\pm + (\bar{z}_\mp \nabla) \bar{z}_\pm = -\nabla p \]

where \( \mathbf{V}_A = \frac{8eB_0}{\eta H_0} \)

1. Alfvén waves moving up the field \( (\mathbf{V} = \frac{8eB}{\eta H_0}) \) have \( \bar{z}^- = 0, \bar{z}^+ \neq 0 \)
2. " " " moving down the field \( (\mathbf{V} = \frac{8eB}{\eta H_0}) \) have \( \bar{z}^+ = 0, \bar{z}^- \neq 0 \)
3. Nonlinear interactions only occur when \( \bar{z}^+ \neq 0 \) and \( \bar{z}^- \neq 0 \) (This requires Alfvén waves moving both directions along \( B_0 \)).

B. I-K Theory

1. Before

\[ \mathbf{V}_A \rightarrow \mathbf{V}_A \rightarrow \mathbf{V}_A \]

a. Interaction time is \( \tau_c = \frac{2}{\mathbf{V}_A} \) (collision time)

2. After

a. Fractional Change in \( \mathbf{V}_e \bar{e} \)
\[ \frac{\Delta \mathbf{V}_e}{\mathbf{V}_e} = \left( \frac{d\mathbf{V}_e}{dt} \tau_c \right) \frac{1}{\mathbf{V}_e} \]

b. \[ \frac{\partial \bar{z}_\pm}{\partial t} \sim (\bar{z}_\mp \nabla) \bar{z}_\pm \sim \frac{\mathbf{V}_e^2}{2} \] \[ \Rightarrow \frac{d\mathbf{V}_e}{dt} \sim \frac{\mathbf{V}_e^2}{2} \]

c. Thus \( \frac{\Delta \mathbf{V}_e}{\mathbf{V}_e} = \left( \frac{\mathbf{V}_e^2}{2} \frac{1}{\mathbf{V}_A} \right) \frac{1}{\mathbf{V}_e} \sim \frac{\mathbf{V}_e}{\mathbf{V}_A} \ll 1 \) (Here we assume \( \mathbf{V}_e \ll \mathbf{V}_A \))

(d) Thus, it takes many collisions to yield \( \frac{\Delta \mathbf{V}_e}{\mathbf{V}_e} \sim 1 \) (\( \mathbf{V}_e \Rightarrow \mathbf{V}_A \))
3. a. Each collision gives a small \( \frac{\Sigma e}{\Sigma v} \) perturbation
   b. Successive collisions add up randomly
   c. \[ N_e \sim \left( \frac{V_e}{\Sigma v} \right)^2 \sim \left( \frac{V_k}{V_e} \right)^2 \]

4. Cascade Time: \[ T_c \sim N_e T_c \sim \left( \frac{V_k}{V_e} \right)^2 \left( \frac{L}{V_k} \right) \sim \frac{L V_k}{V_e^2} \]

5. Cascade Rate: \[ \epsilon = \frac{V_e^2}{T_c} \sim \frac{V_e^2}{T_c} \sim \frac{V_e^4}{L V_k} \]

\[ \Rightarrow V_k \sim \epsilon_0 V_k^{\frac{4}{5}} L^{\frac{4}{5}} \]

6. Predicted 1-D Energy Spectrum: \[ E_k \propto \frac{V_e^2}{k} \sim \epsilon_0 V_k^{\frac{4}{5}} L^{\frac{4}{5}} \frac{V_k}{k} \]

\[ E_k \propto k^{-3/2} \]

Iroshnikov-Kraichnan Spectrum

NOTE: This spectrum is isotropic. The direction of the mean magnetic field plays no role.

III. Anisotropic MHD Turbulence

A. 1. The IK prediction did not match numerical simulations which showed that energy is preferentially transferred to small perpendicular scale \( L_\perp \) rather than small parallel scale \( L_\parallel \).

2. In 1994, Sridhar and Goldreich proposed a model for weak anisotropic MHD turbulence that incorporated anisotropy with respect to the mean field direction.
   a. The original form of this weak turbulence theory was somewhat flawed, but refinements have improved the model.
B. Weak Turbulence

1. If we take $V_e \ll V_A$, the turbulence is weak, meaning it takes many collisions of Alfvén wave packets before energy is transferred nonlinearly from scale $L$ to scale $L/2$.

2. The small corrections $\frac{V_e}{V_A}$ may be treated with perturbation theory.

3. Resonant 3-Wave Interactions:
   a. The dominant nonlinear term in perturbation theory is due to 3-wave interactions: $k_1, k_2, k_3$
   b. Conservation of Momentum requires: $k_1 + k_2 = k_3$
   c. Conservation of Energy requires: $\omega_1 + \omega_2 = \omega_3$
   d. But (defining $\omega > 0$), the Alfvén wave has $\omega = \sqrt{k_1 V_A}$

$$|k_{11}| + |k_{12}| = |k_{13}|$$

e. Taking $k_{11} \geq 0$ and $k_{12} \leq 0$ (colliding waves), we must satisfy both $k_{11} + k_{12} = k_{13}$ and $|k_{11}| + |k_{12}| = |k_{13}|$

f. The only nontrivial solutions have $k_{12} = 0$ and $k_{11} = k_{13}$

$\Rightarrow$ There is no cascade to higher $k_{11}$. Energy is transferred strictly to high $k_1$ in weak turbulence.

4. Collision time:

$$T_c = \frac{L_{11}}{V_A}$$

5. Collision Fractional Change:

$$\frac{Sv_{e_1}}{V_{e_1}} \approx \frac{\partial V_{e_1}}{\partial T_c} T_c V_{e_1} \sim \left(\frac{V_{e_1}}{L_{11}/V_A}\right) T_c \sim \frac{L_{11} V_{e_1}}{L_{11}/V_A}$$
6. Number of Collisions: \( N_{cl} \sim \left( \frac{V_{c1}^2}{V_{c1}} \right) \sim \left( \frac{V_{c1}^2}{V_{c1}^2} \right) \)

7. Cascade Time: \( \tau \sim N_{cl} V_{c1} \sim \left( \frac{V_{c1}^2}{V_{c1}^2} \right) \frac{V_{c1}}{V_{c1}} \sim \frac{V_{c1}^2}{V_{c1}^2} \)

8. Cascade Rate: \( \epsilon \sim \frac{V_{c1}^2}{\tau} = \frac{V_{c1}^2}{\left( \frac{V_{c1}^2}{V_{c1}^2} \right) \frac{V_{c1}}{V_{c1}}} = \epsilon_0^2 \)

\[ \Rightarrow V_{c1} = \epsilon_0 \left( \frac{V_{c1}^2}{V_{c1}^2} \right) \frac{V_{c1}}{V_{c1}} \sim \epsilon_0 \frac{V_{c1}}{V_{c1}} \]

9. Spectrum: \( E_k \sim \frac{V_{c1}^2}{k^2} = \epsilon_0 \frac{V_{c1}^2}{k^2} \sim \epsilon_0 \frac{k}{k^2} \sim \epsilon_0 \frac{1}{k} \)

10. Summary:
   a. Weak Turbulence occurs when \( N_{cl} \gg 1 \Rightarrow V_{c1} \gg V_{c1} \Rightarrow k_{c1} \gg k_{c1} V_{c1} \)

b. No cascade to higher \( k_{c1} \). All turbulence cascades only to higher \( k_{c1} \). \( \Rightarrow \) Anisotropic Cascade in \( k_{c1}, k_{c1} \) Space

   c. 1-D Energy Spectrum: \[ E_k \sim \frac{1}{k^2} \]

   d. Strengthening of the Cascade:

   1. From above, \( V_{c1}^2 \sim \epsilon_0 \frac{V_{c1}^2}{V_{c1}^2} \frac{V_{c1}^2}{V_{c1}^2} \), so \( N_{cl} \sim \left( \frac{V_{c1}^2}{V_{c1}^2} \right) \frac{V_{c1}^2}{V_{c1}^2} \frac{V_{c1}^2}{V_{c1}^2} \)

   2. Thus \( N_{cl} \sim \frac{1}{k_{c1}} \left( \frac{V_{c1}^2}{\epsilon_0 \frac{V_{c1}^2}{V_{c1}^2}} \right) \rightarrow 1 \) as cascade increases \( k_{c1} \).

   3. Thus, nonlinear interactions strengthen until \( N_{cl} \rightarrow 1 \)

\[ \Rightarrow \frac{V_{c1}}{V_{c1}} \sim \frac{V_{c1}}{V_{c1}} \Rightarrow k_{c1} V_{c1} \sim k_{c1} V_{c1} \Rightarrow \text{Critical Balance.} \]
C. Strong MHD Turbulence

(See III.C of Lecture #24)

\[ \log(k \perp) \]

\[ \begin{array}{c}
\text{ Weak Turbulence } \\
\text{ Critical Balance } \\
\text{ GS Case of turbulence } \\
\text{ Strong Turbulence fills this } \\
\end{array} \]