

29/1/95

Homes ①

Lecture #3 Perpendicular Waves and Physical Investigation of Wave Properties

I. Perpendicular Propagating Waves ($\underline{k} \perp \underline{E}$) (Continued)

A. 1. We saw last lecture there are two solutions.

a. $n^2 = P \Rightarrow$ Ordinary Mode (Modified Light Wave) (O)

b. $n^2 = \frac{RL}{S} \Rightarrow$ We'll see this is the Extraordinary Mode (X)

B. Extraordinary Mode: ($n^2 = \frac{RL}{S}$)

1. NOTE: $S - n^2 = S - \frac{RL}{S} = S - \frac{(S+D)(S-D)}{S} = \frac{1}{S} [S^2 - (S^2 - D^2)] = \frac{D^2}{S}$

a. This uses $R = S+D$ & $L = S-D$

b. Thus, we get

$$\begin{pmatrix} S & -iD & 0 \\ iD & \frac{D^2}{S} & 0 \\ 0 & 0 & P - \frac{RL}{S} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

2. Solving for Eigenfunction: a. Take $E_x = E_0$ & $E_z = 0$.

b. $S E_0 - iD E_y = 0 \Rightarrow E_y = \frac{-iS}{D} E_0$

c. So, the eigenfunction is $\underline{E}_1 = (E_0, \frac{-iS}{D} E_0, 0)$

3. Both Longitudinal and Transverse Components:

a. NOTE: $\underline{k} \times \underline{E}_1 = 0$ implies longitudinal fluctuation

$\underline{k} \cdot \underline{E}_1 = 0$ implies transverse fluctuation.

b. For perpendicular propagation let's take $\underline{k} = k \hat{z}$

c. Thus $\underline{k} \times \underline{E}_1 = k \hat{z} \times (E_0 \hat{x} - \frac{iS}{D} E_0 \hat{y}) = -\frac{iS}{D} k E_0 \hat{z} \neq 0$

$$\underline{k} \cdot \underline{E}_1 = k \hat{z} \cdot (E_0 \hat{x} - \frac{iS}{D} E_0 \hat{y}) = k E_0 \neq 0$$

d. This mode is more complicated with both longitudinal and transverse components.

Lecture #3 (Continued)

I. B. (Continued)

4. ~~4.~~ Solution for Frequency: $n^2 = \frac{RL}{S}$

a. After some algebra, this can be written

$$\frac{k^2 c^2}{\omega^2} = \frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{(\omega^2 - \omega_{UH}^2)(\omega^2 - \omega_{LH}^2)}$$

where, as before, $\omega_L = \frac{k|v_{ce}|}{2} + \frac{1}{2} \sqrt{\omega_{ce}^2 + 4(\omega_{pe}^2 + |\omega_{ce}| \omega_{ci})}$

$$\omega_R = \frac{|\omega_{ce}|}{2} + \frac{1}{2} \sqrt{\omega_{ce}^2 + 4(\omega_{pe}^2 + |\omega_{ce}| \omega_{ci})}$$

(Giving $\omega_R = \omega_L + |\omega_{ce}|$, NOTE ω_R, ω_L are solutions of ~~the equation~~
 $\omega^2 - |\omega_{ce}| \omega - \omega_{pe}^2 - |\omega_{ce}| \omega_{ci} = 0$)

b. We have defined the new frequencies:

NOTE! DEF: Upper hybrid frequency: $\omega_{UH}^2 = \omega_{pe}^2 + \omega_{ce}^2$

$\omega_{UH} > \omega_{LH}$

and

DEF: Lower hybrid frequency: $\omega_{LH}^2 = \omega_{ci} |\omega_{ce}| \frac{\omega_{pi}^2}{\omega_{pi}^2 + \omega_{ci} |\omega_{ce}|}$

$\omega_{UH} > \omega_L$

where, for $\omega_{pi}^2 \gg \omega_{ci} |\omega_{ce}|$, $\omega_{LH}^2 \approx \omega_{ci} |\omega_{ce}|$

c. NOTE! In these definitions, we have made use of $\omega_{pe}^2 > \omega_{ce}^2 \gg \omega_{ci}^2$ to simplify.

5. Cutoffs & Resonances of Extraordinary mode $\omega_R > \omega_{UH} > \omega_L > \omega_{LH}$

a. $n^2 > 0$ when $\omega > \omega_R$, $\omega_L < \omega < \omega_{UH}$, $\omega < \omega_{LH}$

$n^2 < 0$ (evanescent) when $\omega_{UH} < \omega < \omega_R$, $\omega_{LH} < \omega < \omega_L$

b. Resonances at: $\omega = \omega_{UH}$ Upper hybrid resonance
 $(n^2 \rightarrow \infty)$ $\omega = \omega_{LH}$ Lower hybrid resonance

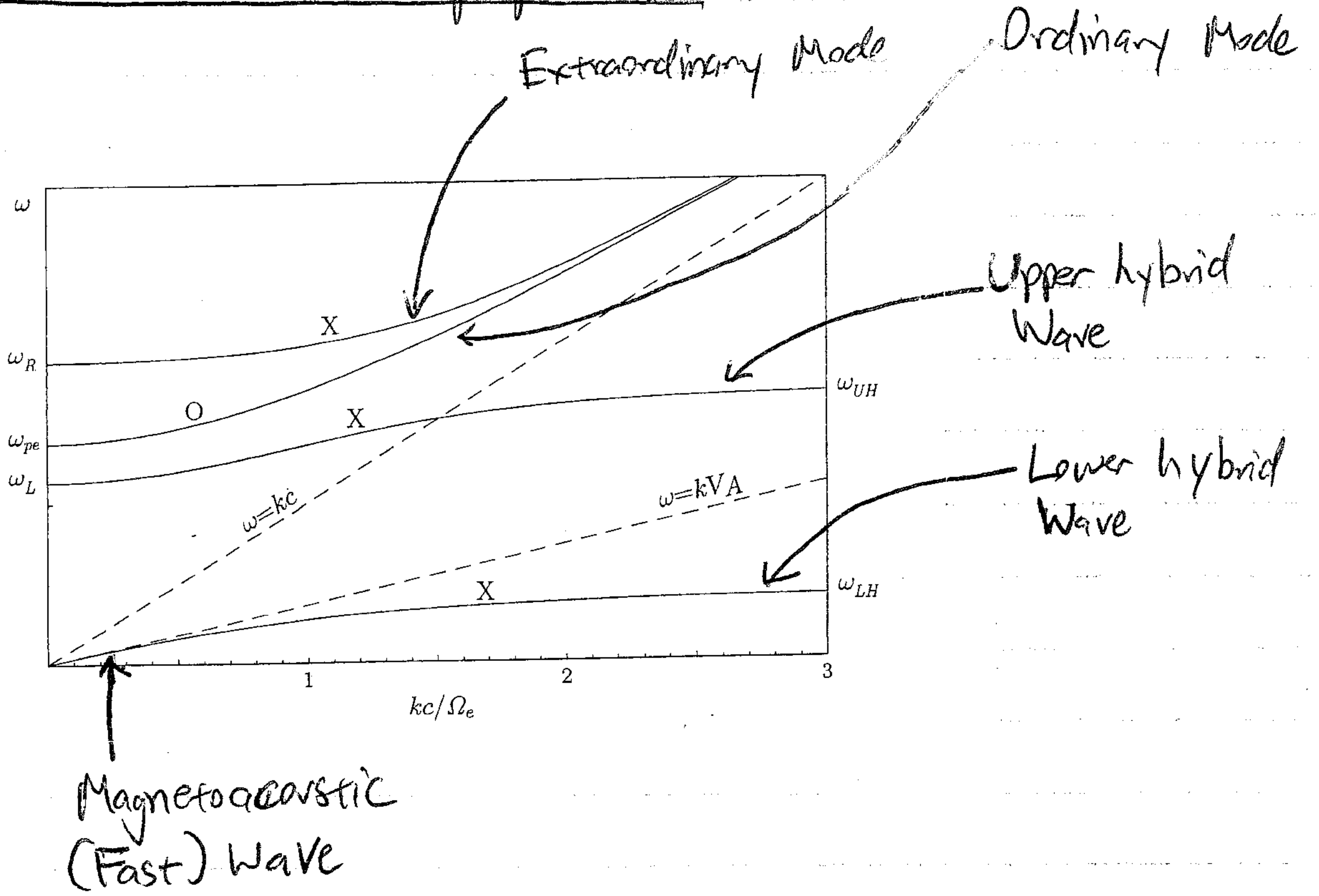
c. Cutoffs at $\omega = \omega_R$
 $(n^2 \rightarrow 0)$ $\omega = \omega_L$

Lecture #3 (Continued)

I. (Continued)

C. Summary of Perpendicular Modes

1. $\omega(k)$ for modes with perpendicular $\underline{k} = k \hat{x}$



II. Summary of Parallel and Perpendicular Waves in a Cold, Magnetized Plasma

A. Parallel

1. $P=0$: **Plasma Oscillation**, longitudinal, motion \parallel to B 2. $n^2 = R$: RH Circularly Polarized Transverse Wave $n^2 = L$: LH Circularly Polarized Transverse WaveLimits: a. **Alfvén Waves** (L, R) $\omega \ll \omega_{ci} \ll |\omega_{ce}| \ll \omega_{pe}$ b. **Ion Cyclotron Waves** (L) $\omega \approx \omega_{ci}$ c. **Whistler Waves** (R) $\omega_{ci} \ll \omega \ll |\omega_{ce}|$ d. **Electron Cyclotron Waves** (R) $\omega \approx |\omega_{ce}|$ e. **LH & RH Circularly Polarized Modified Light Waves** (L, R) $\omega^2 \approx k^2 c^2 + \omega_{pe}^2$

B. Perpendicular

1. $n^2 = P$: **Ordinary Mode**, Modified Light Wave $\omega^2 = c^2 k^2 + \omega_{pe}^2$ 2. $n^2 = \frac{RL}{S}$: **Extraordinary Mode**, longitudinal & transverseLimits: a. **Magnetoacoustic (Fast) Wave** $\omega \ll \omega_{UH}$ b. **Lower Hybrid Waves** $\omega \approx \omega_{LH}$ c. **Upper Hybrid Waves** $\omega \approx \omega_{UH}$ d. **Extraordinary Mode Modified Light Wave** $\omega^2 \approx k^2 c^2 + \omega_{pe}^2$

III. The Lower Hybrid Wave

A. Limits of the Lower Hybrid Wave ($\mathbf{E} \perp \mathbf{B}_0$)

1. Take $\mathbf{k} = k \hat{x}$ $\mathbf{B}_0 = B_0 \hat{z}$ (we can also write $\hat{b} = \frac{\mathbf{B}}{B} = \hat{z}$)
2. This Extraordinary Mode has $\mathbf{E}_1 = (E_0, -\frac{iS}{D} E_0, 0)$
3. $\omega_{LH}^2 = \omega_{ci} |\omega_{ce}|$ (assuming $\omega_{pi}^2 \gg \omega_{ci} |\omega_{ce}|$)
4. We'll investigate the wave behavior in the limit $k \rightarrow \infty$.
5. For simplicity, assume a proton & electron plasma: $q_e = -q_i$, $\frac{m_i}{m_e} = 1836$.

B. Ion Current:

1. Eq. of Motion (Momentum Eq.): $\mathbf{U}_i = \underbrace{\frac{q_i}{-i\omega m_i} \mathbf{E}_1}_{\text{Electric field term}} + \underbrace{\frac{q_i B_0}{-i\omega m_i} \mathbf{U}_i \times \hat{b}}_{\text{Lorentz Force term}}$

2. $\mathbf{U}_i = i \frac{q_i}{\omega m_i} \mathbf{E}_1 + i \frac{\omega_{ci}}{\omega} \mathbf{U}_i \times \hat{b}$

① ② ③

3a. Let's compare the magnitudes of ③ for $\omega = \omega_{LH}$

b. NOTE: For Lower Hybrid $\omega^2 = \omega_{ci} |\omega_{ce}| = \frac{q_i B_0}{m_i} \frac{k \ell B_0}{m_e} = \frac{q_i^2 B_0^2}{m_i^2} \frac{m_i}{m_e} = \omega_{ci}^2 \frac{m_i}{m_e}$

c. Thus, $\frac{\omega_{ci}}{\omega} = \sqrt{\frac{m_e}{m_i}} \ll 1$

c. Thus $\mathcal{O}\left(\frac{③}{①}\right) = \frac{\omega_{ci} \ell}{\omega} = \frac{\omega_{ci}}{\omega} \ll 1$

\Rightarrow Lorentz Force term ③ is negligible: Ions are unmagnetized!

4. Ion current: $\mathbf{j}_i = n_i q_i \mathbf{U}_i = \frac{i \epsilon_0 n_i q_i^2}{\omega \epsilon_0 m_i} \mathbf{E}_1 = \boxed{i \epsilon_0 \frac{\omega_{pi}^2}{\omega} \mathbf{E}_1 = \mathbf{j}_i}$

C. Electron Current:

1. Eq. of Motion: $\mathbf{U}_e = \frac{q_e}{-i\omega m_e} \mathbf{E}_1 + \frac{q_e B_0}{-i\omega m_e} \mathbf{U}_e \times \hat{b}$

2. $\mathbf{U}_e = i \frac{q_e}{\omega m_e} \mathbf{E}_1 + i \frac{\omega_{ce}}{\omega} \mathbf{U}_e \times \hat{b}$

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3. NOTE: $\omega^2 = \omega_{ci} |\omega_{ce}|^2 = \frac{q_e^2 B_0^2}{m_e^2} \frac{m_e}{m_i} \Rightarrow \frac{\omega_{ce}}{\omega} = \sqrt{\frac{m_i}{m_e}} \gg 1$.

4. Lorentz Force Dominates: Electrons are magnetized!

\Rightarrow Larmor motion is lowest order motion

III C. (Continued)

5. We can use our knowledge of single particle motion \Rightarrow drifts.

$$\underline{v}_e = \underbrace{\frac{iq_e}{\omega m_e} E_z \hat{b}}_{\text{acceleration along } \underline{B} \text{ field}} + \underbrace{\frac{\underline{E}_1 \times \hat{b}}{B_0}}_{\text{E} \times \underline{B} \text{ drift (Lect #3 29:194)}} - \underbrace{\frac{i\omega}{\omega_{ce} B_0} \underline{E}_1}_{\text{Polarization Drift (Lect #9 29:194)}}$$

$E_z = 0$
for lower hybrid

6. Electron current: $\underline{j}_e = ne q_e \underline{v}_e = \epsilon_0 \left(\frac{ne q_e^2}{\epsilon_0 m_e} \right) \frac{(me)}{q_e B_0} \underline{E}_1 \times \hat{b} - \frac{iq_e \omega}{\omega_{ce}} \left(\frac{ne q_e^2}{\epsilon_0 m_e} \right) \frac{(me)}{q_e B_0} \underline{E}_1$

$$\underline{j}_e = \epsilon_0 \frac{\omega_{pe}^2}{\omega_{ce}} \underline{E}_1 \times \hat{b} - i \epsilon_0 \frac{\omega \omega_{pe}^2}{\omega_{ce}^2} \underline{E}_1$$

D. Limiting Behavior of Lower Hybrid as $k \rightarrow \infty$

1. Faraday's Law: $\underline{k} \times \underline{E}_1 = \omega \underline{B}_1$ (A)

2. Ampere/Maxwell Law: $\underline{k} \times \underline{B}_1 = -i \mu_0 \underline{j} - \frac{\omega}{c^2} \underline{E}_1$

$$\Rightarrow c^2 \underline{k} \times \underline{B}_1 = \underbrace{\frac{\omega \mu_0^2}{\omega} \underline{E}_1}_{\text{ion current}} - i \underbrace{\frac{\omega_{pe}^2}{\omega_{ce}} \underline{E}_1 \times \hat{b}}_{\text{E} \times \underline{B} \text{ electron current}} - \underbrace{\frac{\omega \omega_{pe}^2}{\omega_{ce}^2} \underline{E}_1}_{\text{electron polarization current}} - \omega \underbrace{\underline{E}_1}_{\text{displacement current}} \quad \text{(B)}$$

3. Lowest order solution for $k \rightarrow \infty$

a. From (A) $\underline{k} \times \underline{E}_{(0)} = 0 \Rightarrow \underline{E}_{(0)} = E_0 \hat{k}$ Electrostatic

NOTE: This can also be seen from eigenfunction: $\underline{E}_1 = (E_0, \frac{iS}{D} E_0, 0)$

As $k \rightarrow \infty$, $S \rightarrow 1$, $D \rightarrow \infty$, so $\frac{E_y}{E_x} \rightarrow 0$.

b. From (B) $\underline{k} \times \underline{B}_{(0)} = 0 \Rightarrow \underline{B}_{(0)} = B_1 \hat{k}$. But $\underline{k} \cdot \underline{B}_1 = 0$, so $B_1 = 0$!
No magnetic response!

4. Next order: Using (B) with $\underline{E}_{(0)} = E_0 \hat{k}$, we get

$$c^2 \underline{k} \times \underline{B}_{(1)} = \frac{\omega \mu_0^2}{\omega} E_0 \hat{k} - i \frac{\omega_{pe}^2}{\omega_{ce}} E_0 \hat{k} \times \hat{b} - \frac{\omega \omega_{pe}^2}{\omega_{ce}^2} E_0 \hat{k} - \omega E_0 \hat{k}$$

III D. (Continued)

5. Take $\hat{k} \cdot \textcircled{B}$ to eliminate $B_{\parallel}(t)$:

$$a. 0 = \frac{\omega_{pi}^2}{\omega} E_0 - \frac{\omega \omega_{pe}^2}{\omega_{ce}^2} E_0 - \omega E_0$$

b. This can be solved to yield the Lower Hybrid Frequency

$$\omega^2 = \omega_{ci} \omega_{ce} \left(\frac{\omega_{pi}^2}{\omega_{pi}^2 + \omega_{ci} \omega_{ce}} \right)$$

E. Physics of the Lower Hybrid Wave

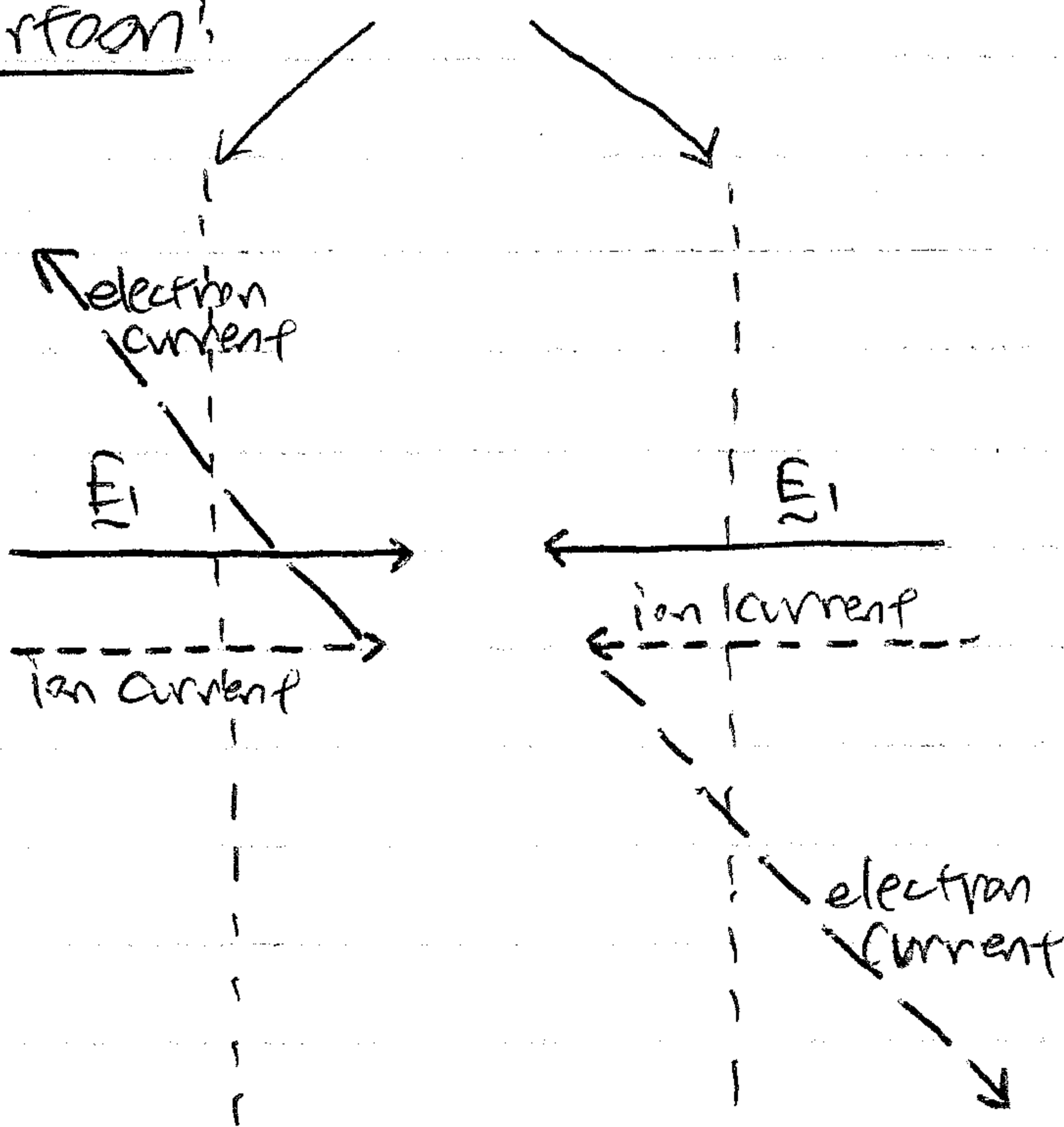
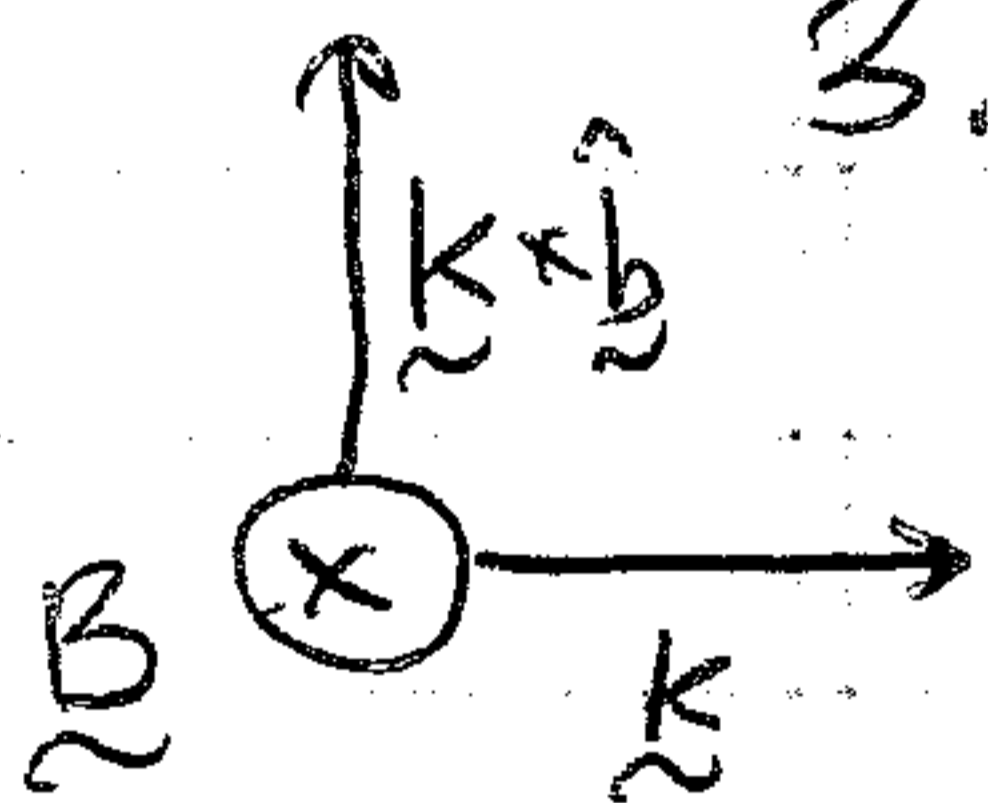
1. The current along \hat{k} ($\perp B_0$) sums to zero

- a. This is from
 - i. ion current (ion magnetized)
 - ii. electron polarization current
 - iii. displacement current

2. $E \times B$ electron current suppresses (small) B_{\parallel} fluctuations.

3. Cartoon!

Wave crests



No charge build up:

$$\nabla \cdot \vec{j} = 0$$

$k \cdot \vec{j} = 0$ currents in \hat{k} direction cancel.

IV. Homework Assignment: Cold Plasma Wave Presentations

A. Choose one of the following waves with the specified limits.

- | | | |
|-----------------------------------|--|------------------------|
| 1. Cold Plasma Alfvén Waves | $\omega \ll \omega_{ci} \ll \omega_{ce} \ll \omega_{pe}$ | $k \rightarrow 0$ |
| 2. Ion Cyclotron Waves | $\omega \approx \omega_{ci}$ | $k \rightarrow \infty$ |
| 3. Whistler Waves | $\omega_{ci} \ll \omega \ll \omega_{ce} \ll \omega_{pe}$ | $k \rightarrow \infty$ |
| 4. Electron Cyclotron Waves | $\omega \approx \omega_{ce}$ | $k \rightarrow \infty$ |
| 5. Upper Hybrid Waves | $\omega^2 \approx \omega_{pe}^2 + \omega_{ce}^2$ | $k \rightarrow \infty$ |
| 6. Extraordinary Mode Light Waves | $\omega > \omega_R$ | |

B. Present the following

1. Limits of Wave (Setup of k , B , etc.)
2. Ion and Electron current (are they magnetized?)
3. Limiting Behavior
4. Solution of Mode Frequency
5. Physical Description
6. Cartoon of Mechanism (this is important)

NOTE: Please produce a 1-2 page summary for distribution as notes for the class. You do not need to go through all of the steps, but please do outline the path you followed.