29:195 Howes ①

Lecture #6: Ray Tracing in Inhomogeneous Plasmas

I. Introduction

A. Inhomogeneous Plasmas

1. Everything we have learned about waves so far has been for homogeneous plasmas ⇒ we can Fourier analyze to solve.
2. In reality, completely homogeneous plasmas do not exist (but we'll see the lowest order properties of waves in inhomogeneous plasmas corresponds to the homogeneous solution).

3. Ray Tracing
   - Ray tracing is a technique used to solve for fields in many physical situations:
     a. Radio waves in plasmas
     b. Propagation of seismic waves in the earth & the sun
     c. General relativistic bending of light by gravity in galaxy clusters

B. Wave Propagation in an Inhomogeneous, Cold, Unmagnetized Plasma

1. For simplicity, we'll consider a cold, unmagnetized plasma with an equilibrium density gradient $n_0 = n_0(\underset{\sim}{x}, t)$

   a. Since $\omega_p^2(\underset{\sim}{x},t) = \frac{n_0(\underset{\sim}{x},t)}{\epsilon_0}\left(\frac{q_i^2}{m_i} + \frac{q_e^2}{m_e}\right)$ (assuming $n_{0i} = n_{0e}$)

   ⇒ The plasma frequency changes in space and time.

2. From Lect #22 of 29:194 (Eq III.C.3.b.)

   a. $c^2 \underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}_1) = \omega_p^2 \underset{\sim}{E}_1 - \omega^2 \underset{\sim}{E}_1$ where $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$

   b. We can go through all the same steps <u>without</u> Fourier transforming to get:

$$\boxed{-c^2 \nabla\times(\nabla\times\underset{\sim}{E}_1) = \omega_p^2(\underset{\sim}{x},t)\,\underset{\sim}{E}_1 + \frac{\partial^2 \underset{\sim}{E}_1}{\partial t^2}} \quad (1)$$

From Eq. (8)

$$\frac{1}{\epsilon}\vec{E}_{1(0)} + \frac{1}{\epsilon}\epsilon\vec{E}_{1(1)} = \omega\vec{B}_{1(0)} + \omega\epsilon\vec{B}_{1(1)} \overset{order\frac{1}{\epsilon}}{\rightarrow} \vec{k}\times\vec{E}_1 = 0$$

Analog for $\vec{B_1}$ with Eq. (9)

$$\begin{aligned} \Rightarrow \vec{E}_{1(0)} &= E_0\hat{\vec{k}} \quad (longitudinal) \\ \Rightarrow \vec{B}_{1(0)} &= 0 \end{aligned}$$

3) Eq.(7) in Eq.(6) and $\vec{E_1} = E_0\hat{\vec{k}}$

$$\omega\vec{U}_e\hat{\vec{k}} = i\frac{q_e}{m_e}E_0 + \frac{\omega_{ce}^2}{\omega}\vec{U}_e\hat{\vec{k}} \tag{10}$$

4) Poisson's Eq.:

$$\nabla\vec{E_1} = \frac{\sum_s n_s q_s}{\epsilon_0}$$

gives us (fixed ions; quasi neutrality):

$$\vec{k}\vec{E_1} = kE_0 = -i\frac{q_e n_e}{\epsilon_0} \tag{11}$$

5) Continuity Eq.:

$$\frac{\partial n_e}{\partial t} + \nabla.(n_e\vec{U_e}) = 0$$

gives us:

$$n_e = \frac{\vec{k}\vec{U}_e n_0}{\omega} \tag{12}$$

6) Eq.(10)+Eq.(11)+Eq.(12) gives us:

$$\omega\vec{U}_e\hat{\vec{k}} = \frac{q_e^2 n_0}{m_e\epsilon_0\omega}\vec{U}_e\hat{\vec{k}} + \frac{\omega_{ce}^2}{\omega}\vec{U}_e\hat{\vec{k}} = \frac{\omega_{pe}^2}{\omega}\vec{U}_e\hat{\vec{k}} + \frac{\omega_{ce}^2}{\omega}\vec{U}_e\hat{\vec{k}} \tag{13}$$

7) Eq. (13) can be solved to yield the Upper Hybrid Frequency:

$$\boxed{\omega^2 = \omega_{pe}^2 + \omega_{ce}^2}$$

E) Physics of the Upper Hybrid Wave:

1)An electric field in the x-direction ($\hat{\vec{k}}$-direction) will accelerate the electrons in the negative x- direction displaced from their rest position. As the electrons pick up speed, the Lorentz force will become larger and turn the electrons toward the positive y-direction ($\hat{\vec{b}} \times \hat{\vec{k}}$). Eventually, the electrons are turned around and move against the electric field, losing energy (elliptical orbit). Thus two restoring forces act on the electrons: the electrostatic force resulting from the electrons displacement and the Lorentz force. This additional restoring force leads to a higher frequency than in simple plasmas oscillations. If the magnetic field vanishes, the cyclotron frequency vanishes, too, leaving us with ordinary plasma oscillations. If the plasma density decreases, the plasma frequencey decreases, too. For vanishing plasma density, the remaining motion is a gyration around the magnetic field line.

2)Because of the high frequency and the big mass of ions the ion current is negligible. The heavy ions can't follow the rapidly changing electric field.

3) $\vec{E}\times\vec{B}$ electron current supports small $\vec{B_1}$ fluctuations.
4)Cartoon:

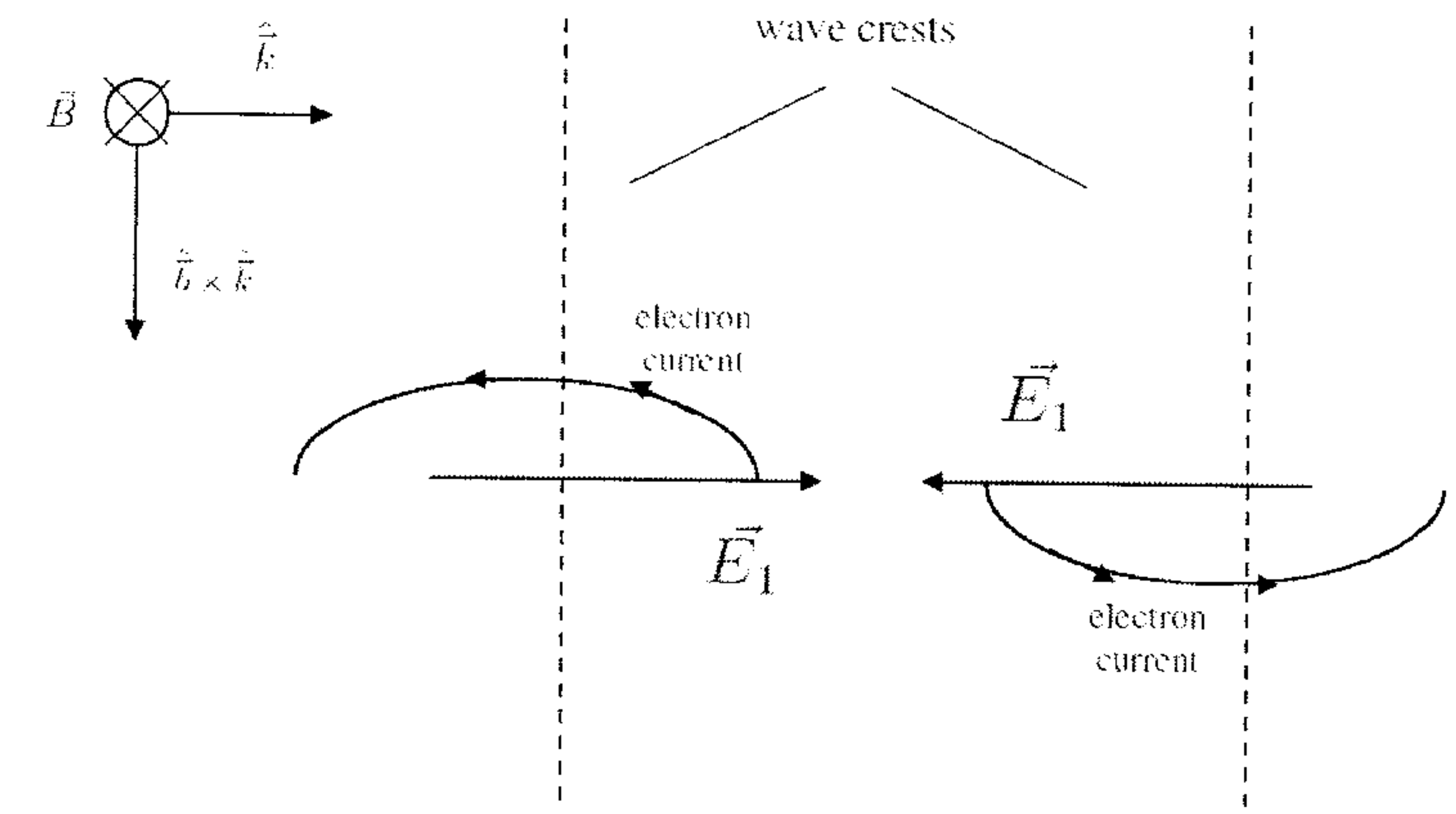

Figure 1: Elliptical Electron Trajectory

THE ELECTRON CYCLOTRON WAVE • p^+e^- plasma

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I) LIMITS

• $\bar{K} \parallel \bar{B}$ ($\bar{K} = K\hat{z}$; $\bar{B}_o = B_o\hat{z}$) • Eigen function: $\bar{E}_1 = (E_o, iE_o, 0)$

• Transverse: $\bar{E}_1 \times \bar{K} \neq 0$ • No Longitudinal: $\bar{E}_1 \cdot \bar{K} = 0$ • $\omega \simeq \omega_{ce}$; $K \to \infty$

II) CURRENTS

A) Ions : EoM:
$$\underset{(i)}{\bar{U}_i} = \underset{(ii)}{\frac{iq_i}{\omega m_i}\bar{E}_1} + \underset{(iii)}{\frac{iq_iB_o}{\omega m_i}\bar{U}_i \times \hat{b}}$$

$$\mathcal{O}\left(\frac{iii}{i}\right) = \left(\frac{\frac{\omega_{ci}}{\omega}\bar{U}_i \times \hat{b}}{\bar{U}_i}\right) = \frac{\frac{q_iB_o}{m_i}}{\frac{|q_e|B_o}{m_e}} = \frac{m_e}{m_i} << 1$$

$\boxed{\text{Ions Unmagnetized}}$

$-i\omega\bar{U}_i = \frac{q_i}{m_i}\bar{E}_1$; as $\omega \to$ Large, $\bar{U}_i \to 0$.

$\boxed{\text{Ions Immobile}}$

B) Electrons : EoM:
$$\underset{i)}{\bar{U}_e} = \underset{ii)}{\frac{iq_e}{\omega m_e}\bar{E}_1} + \underset{iii)}{\frac{i\epsilon\omega_{pe}^2}{\omega_{ce}^2}\bar{E}_1 \times \hat{b}} + \underset{iv)}{\frac{1}{\omega_{ce}B_o}\frac{d}{dt}\bar{E}_1}$$

$$\mathcal{O}\left(\frac{iii}{i}\right) = \left(\frac{\frac{\omega_{ce}}{\omega}\bar{U}_e \times \hat{b}}{\bar{U}_e}\right) \simeq \frac{\omega_{ce}}{\omega_{ce}} = 1 \quad \boxed{\text{Electrons Magnetized}}$$

$$\bar{J}_e = \frac{-i\bar{E}_1}{B_o} + \epsilon_o\frac{\omega_{pe}^2}{\omega_{ce}^2}\bar{E}_1 \times \hat{b} + \frac{1}{\omega_{ce}B_o}\frac{d}{dt}\bar{E}_1$$

$$\boxed{\bar{J}_e \simeq \epsilon_o\frac{\omega_{pe}^2}{\omega_{ce}^2}\bar{E}_1 \times \hat{b} - \frac{i\bar{E}_1}{B_o}}$$

III) SOLUTION OF MODE FREQUENCY

$$n^2 = R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} = 1 - \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega+\omega_{ci})}$$

$$\Rightarrow \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})} + \frac{\omega_{pi}^2}{\omega(\omega+\omega_{ci})} = 1 - \frac{K^2c^2}{\omega^2}$$

$$1 - \frac{K^2c^2}{\omega^2} \simeq \frac{\omega_{pe}^2 + \omega_{pi}^2}{2\omega_{ce}^2} = \frac{\omega_p^2}{2\omega_{ce}^2}$$

Rearranging, we have:
$$\Rightarrow 2\omega_{ce} = \frac{\omega_p^2}{\omega_{ce}}\left(1 - \frac{K^2c^2}{\omega^2}\right)^{-1}$$

$$\Rightarrow \boxed{|\omega - \omega_{ce}| = \frac{\omega_p^2}{\omega_{ce}}\left(1 - \frac{K^2c^2}{\omega^2}\right)^{-1}}$$

; as $2\omega_{ce} \simeq \omega + |\omega_{ce}| = |\omega - \omega_{ce}|$

$\bar{K} \times \bar{E}_1 = \omega\bar{B}_1 \neq 0$
$\Rightarrow$ An ElectroMagnetic Wave

$$\bar{K} \times \bar{E}_1 \to \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\frac{E_oK}{\omega} = \bar{B}_1$$

IV LIMITING BEHAVIOR

As $K \to \infty$, $|\omega - \omega_{ce}| = \frac{\omega_p^2}{\omega_{ce}\left(1 - \frac{K^2c^2}{\omega^2}\right)} \to 0$

Thus, for K Large, $\omega = \omega_{ce}$

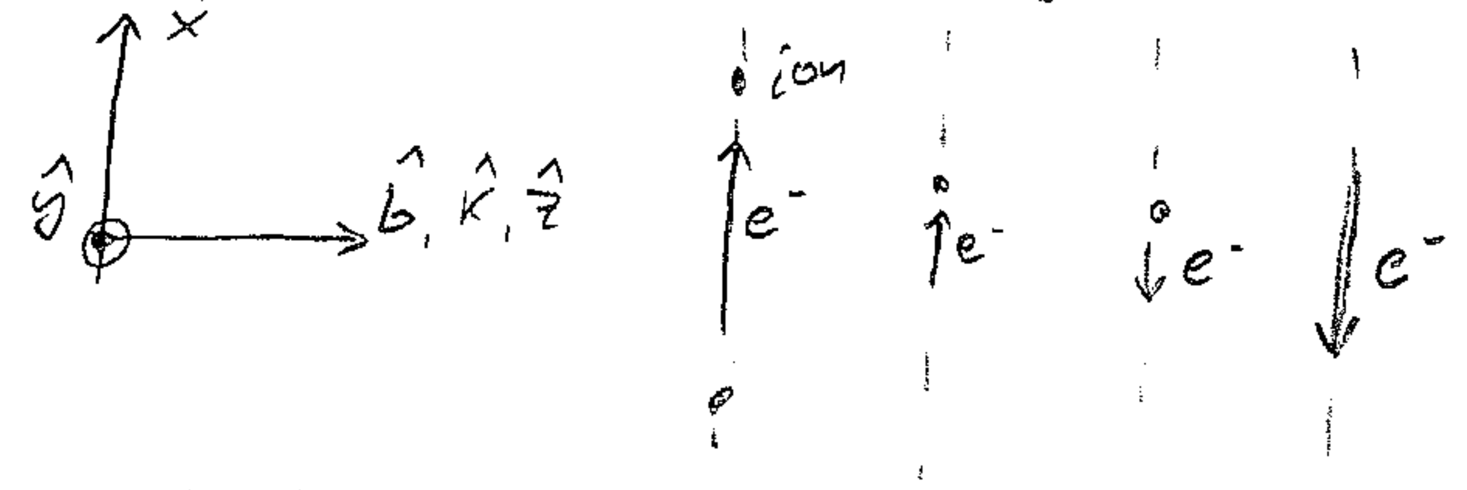
$$V_p = \frac{\omega_p^2}{K\omega_{ce}\left(1 - \frac{K^2c^2}{\omega^2}\right)} + \frac{\omega_{ce}}{K}$$

$$K \to \infty \quad V_p \to 0 + \frac{\omega_{ce}}{K}$$

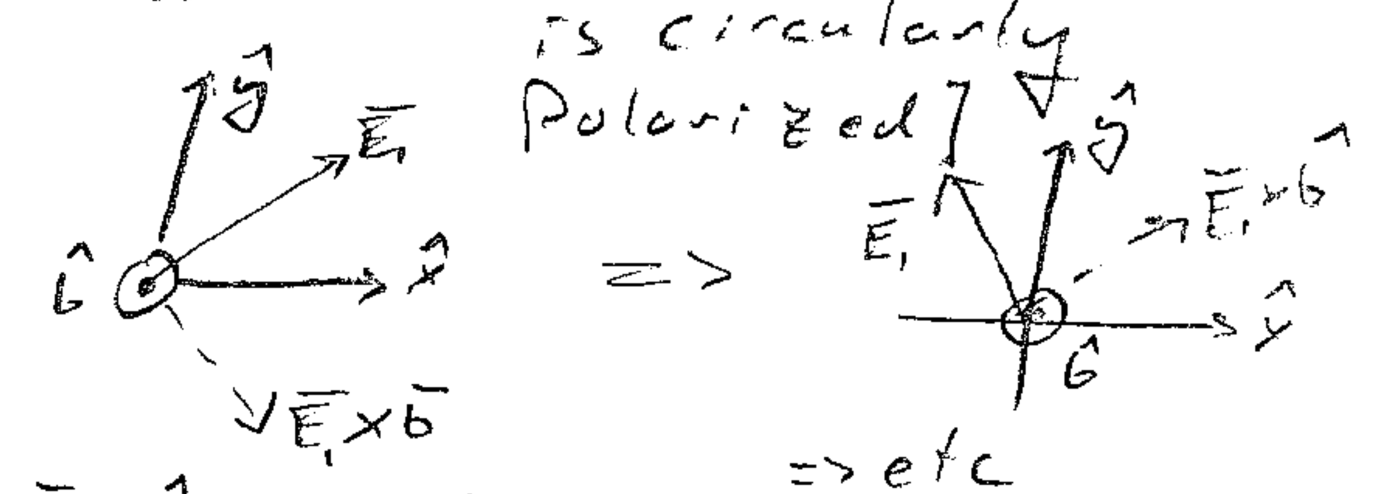
$$V_g = \frac{\partial\omega}{\partial K} = \frac{\partial}{\partial K}\left(\frac{\omega_p^2}{\omega_{ce}}\frac{1}{\left(1 - \frac{K^2c^2}{\omega^2}\right)} + \omega_{ce}\right) = \frac{\omega_p^2}{\omega_{ce}}\frac{2K}{\left(1 - \frac{K^2c^2}{\omega^2}\right)^2} \text{ ; } K \to \infty \quad V_g \to \frac{2\omega_p^2\omega_{ce}^2}{K^3c^4}$$

V PHYSICAL DESCRIPTION

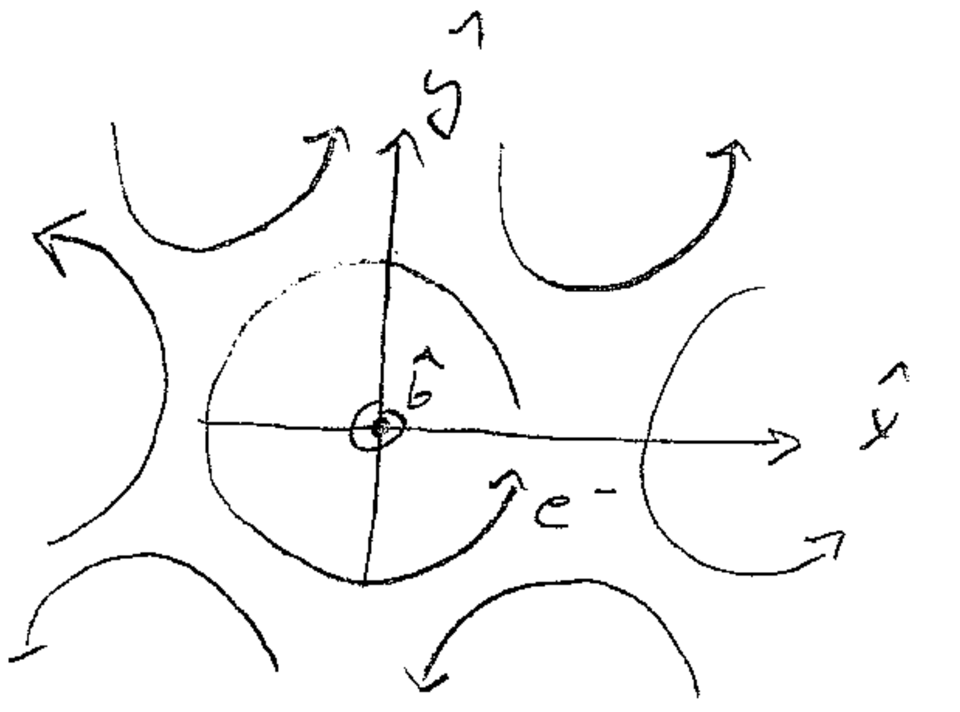
An EM Wave with immobile ions and electrons constricted to moving in the plane $\perp$ to $\hat{b}$. $\bar{J}_e = \epsilon_o\frac{\omega_{pe}^2}{\omega_{ce}^2}\bar{E}_1 \times \hat{b}$ [Recall that $\bar{E}_1$ is circularly Polarized]



As each wave passes, all of the e^- in a given x-y plane will be engaged in the same circular motion



$\bar{E}_1 \times \hat{b}$ lags $\bar{E}_1$ by $\pi/2$



The electrons exhibit circular motion in the x-y plane. For $\omega \simeq \omega_{ce}$, the e^- absorb a significant amount of energy from the wave. Can be used to heat a plasma.

II. B. (Continued)

4. Expand solution $\underset{\sim}{E}_1$ in powers of ϵ: $\underset{\sim}{E}_1 = \underset{\sim}{E}_{1(0)} + \epsilon \underset{\sim}{E}_{1(1)} + \epsilon^2 \underset{\sim}{E}_{1(2)} + \dots$

C. $\mathcal{O}(1)$ Solution:

1. $\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}_{1(0)}) = \frac{\omega_p^2 - \omega^2}{c^2} \underset{\sim}{E}_{1(0)}$
   a. This just gives the dispersion relation for a homogeneous plasma.
   $\Rightarrow$ At lowest order, local conditions $\underset{\sim}{k}(\underset{\sim}{x},t)$ & $\omega(\underset{\sim}{x},t)$ satisfy homogeneous dispersion relation

2. Let's focus on the Modified Light Wave $\Rightarrow$ take $\underset{\sim}{k} \cdot \underset{\sim}{E}_{1(0)} = 0$

$$\Rightarrow \boxed{\omega^2(\underset{\sim}{x},t) = \omega_p^2(\underset{\sim}{x},t) + k^2(\underset{\sim}{x},t)\, c^2}$$

   a. Usually, $\omega = \omega(\underset{\sim}{x}, \underset{\sim}{k}, t)$, but since $\underset{\sim}{k} = \underset{\sim}{k}(\underset{\sim}{x},t)$, we may write $\omega = \omega(\underset{\sim}{x},t)$.

3. a. Assuming we know $n(\underset{\sim}{x},t)$, then $\omega_p^2(\underset{\sim}{x},t)$ is known.
   b. This leaves us with 4 unknowns $[\omega(\underset{\sim}{x},t)$ & $\underset{\sim}{k}(\underset{\sim}{x},t)]$ and <u>one</u> equation.
   c. But, ω & $\underset{\sim}{k}$ are related $\Rightarrow$ Both derived from one function, $s(\underset{\sim}{x},t)$.

4. a. Remember, by definition, $\frac{\partial \underset{\sim}{k}}{\partial t} = -\nabla \omega$
   b. But $\omega = \omega(\underset{\sim}{x}, \underset{\sim}{k}(\underset{\sim}{x},t), t)$ so

$$\nabla \omega = \frac{\partial \omega}{\partial \underset{\sim}{x}} = \left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t} + \left(\frac{\partial \underset{\sim}{k}}{\partial \underset{\sim}{x}}\right) \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t} = \left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t} + \underbrace{\nabla \underset{\sim}{k}}_{\text{Tensor}} \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t}$$

   c. <u>Subtle point</u>: $\nabla \underset{\sim}{k} = \nabla(\nabla s) \Rightarrow$ This is a symmetric tensor, so we may write $\nabla \underset{\sim}{k} \cdot \frac{\partial \omega}{\partial \underset{\sim}{k}} = \frac{\partial \omega}{\partial \underset{\sim}{k}} \cdot \nabla \underset{\sim}{k}$

   d. This gives: $\frac{\partial \underset{\sim}{k}}{\partial t} + \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t} \cdot \nabla \underset{\sim}{k} = -\left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t}$

5. Remember, group velocity $\underset{\sim}{v}_g \equiv \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t}$ $\Rightarrow$ This is the group velocity at a given point $\underset{\sim}{x}$.

II. C. (Continued)

6. Lagrangian Frame:

a. Follow a point moving with group velocity: $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v}_g = \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t}$

b. The Lagrangian (or convective, substantial) derivative is

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{v}_g \cdot \nabla$$

c. Thus $\frac{d\underset{\sim}{k}}{dt} = -\left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t}$

d. $$\frac{d\omega}{dt} = \left(\frac{\partial \omega}{\partial t}\right)_{\underset{\sim}{k},\underset{\sim}{x}} + \cancel{\underbrace{\frac{d\underset{\sim}{k}}{dt}}_{-\frac{\partial \omega}{\partial \underset{\sim}{x}}} \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t}} + \cancel{\underbrace{\frac{\partial \underset{\sim}{x}}{\partial t}}_{=\frac{\partial \omega}{\partial \underset{\sim}{k}}} \cdot \left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t}} = \left(\frac{\partial \omega}{\partial t}\right)_{\underset{\sim}{k},\underset{\sim}{x}}$$

D. The Ray Equations

1.
$$\boxed{\begin{aligned} \frac{d\underset{\sim}{k}}{dt} &= -\left(\frac{\partial \omega}{\partial \underset{\sim}{x}}\right)_{\underset{\sim}{k},t} \\ \frac{d\underset{\sim}{x}}{dt} &= \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{\underset{\sim}{x},t} \\ \frac{d\omega}{dt} &= \left(\frac{\partial \omega}{\partial t}\right)_{\underset{\sim}{x},\underset{\sim}{k}} \end{aligned}}$$

The Ray Equations are completely analogous to Hamilton's equations under the change

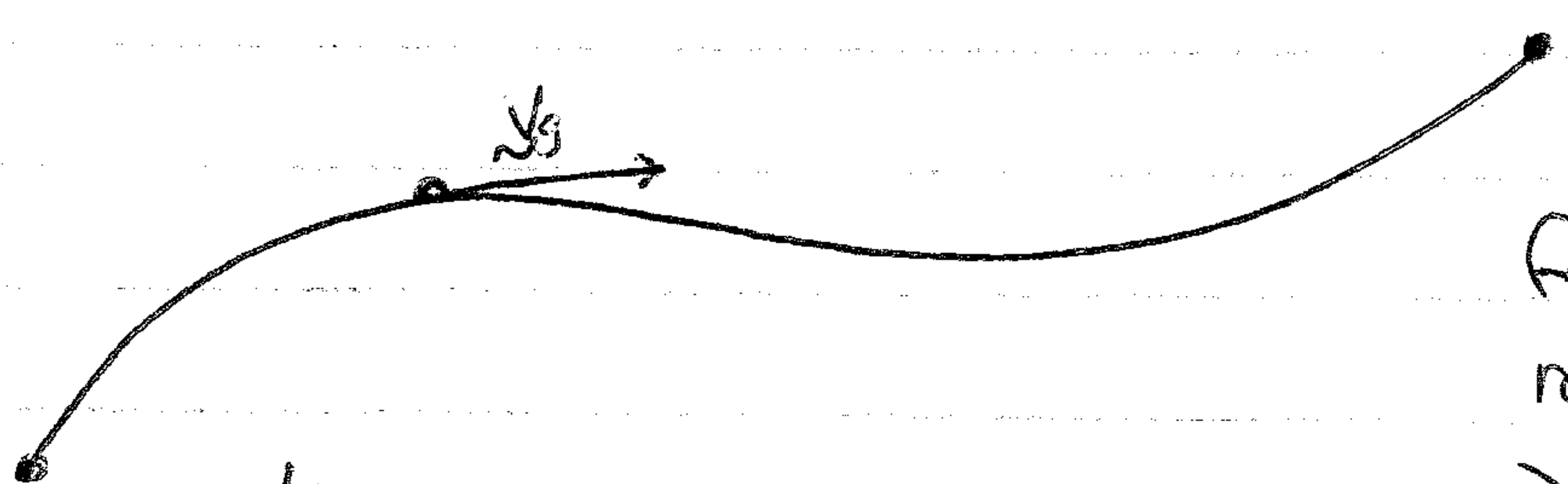
$\omega \Rightarrow H$

$\underset{\sim}{x} \Rightarrow x$

$\underset{\sim}{k} \Rightarrow p$

III. Solving the Ray Equations

A. i.



Beginning of ray: $\underset{\sim}{x}_0, \underset{\sim}{k}_0, \omega_0$ are known

After some time: $\underset{\sim}{x}, \underset{\sim}{k}, \omega$ found by integrating ray equations

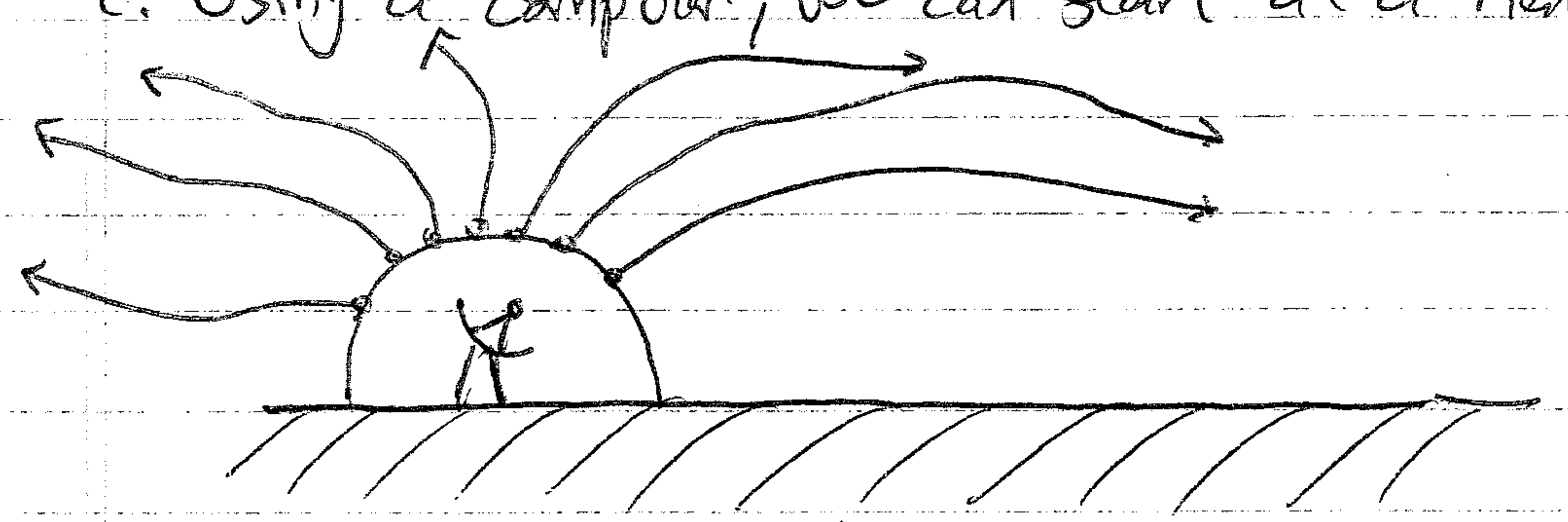
$\Rightarrow$ Just like a particle trajectory.

III. A. (Continued)

2. a. Solving for $S(\underset{\sim}{x},t)$: We can find $S(\underset{\sim}{x},t)$ by integrating along the ray:
$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \underset{\sim}{V_g}\cdot\nabla S = -\omega(\underset{\sim}{x},t) + \underset{\sim}{V_g}\cdot\underset{\sim}{k}(\underset{\sim}{x},t)$$

b. But this only gives S along the ray.

c. Using a computer, we can start at a hemisphere of points:



d. By integrating along many path paths, we can eventually find $S(\underset{\sim}{x},t)$ over all space (by interpolation)

B. Amplitudes:

1. a. Our original solution assumed $\overline{E}_1(\underset{\sim}{x},t) = E_1(\underset{\sim}{x},t)\, e^{iS(\underset{\sim}{x},t)}$

b. We have solved for the eikonal $S(\underset{\sim}{x},t)$, but usually we want to know the amplitude as well.

c. To solve for amplitude, we go to the next order in the expansion.

2. $\mathcal{O}(1)$:

a. $$\underbrace{\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}_{1(1)}) - \frac{\omega_p^2-\omega^2}{c^2}\underset{\sim}{E}_{1(1)}}_{\text{We don't need to know } E_{1(1)}} = \underbrace{i\underset{\sim}{k}\times(\nabla\times\underset{\sim}{E}_{1(0)}) + i\nabla\times(\underset{\sim}{k}\times\underset{\sim}{E}_{1(0)}) - \frac{i\omega}{c^2}\frac{\partial \underset{\sim}{E}_{1(0)}}{\partial t} - \frac{i}{c^2}\frac{\partial}{\partial t}(\omega\underset{\sim}{E}_{1(0)})}_{\text{We want to find } \underset{\sim}{E}_{1(0)}}$$

b. Annihilate $\underset{\sim}{E}_1$ by dotting solution with $\underset{\sim}{E}^*_{1(0)}$:

i. $$\underset{\sim}{E}^*_{1(0)}\cdot\left[\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}_{1(1)}) - \frac{\omega_p^2-\omega^2}{c^2}\underset{\sim}{E}_{1(1)}\right] = \overbrace{\cancel{(\underset{\sim}{E}^*_{1(0)}\cdot\underset{\sim}{k})}}^{0}(\underset{\sim}{k}\cdot\underset{\sim}{E}_{1(1)}) + \overbrace{\cancel{\left(-k^2 - \frac{\omega_p^2-\omega^2}{c^2}\right)}}^{0}\underset{\sim}{E}^*_{1(0)}\cdot\underset{\sim}{E}_{1(1)} = 0$$

c. We may then add the resulting RHS to it's complex conjugate and manipulate.

Lecture #6 (Continued)

III B. (Continued)

3. Continuity Equation for Wave Energy:

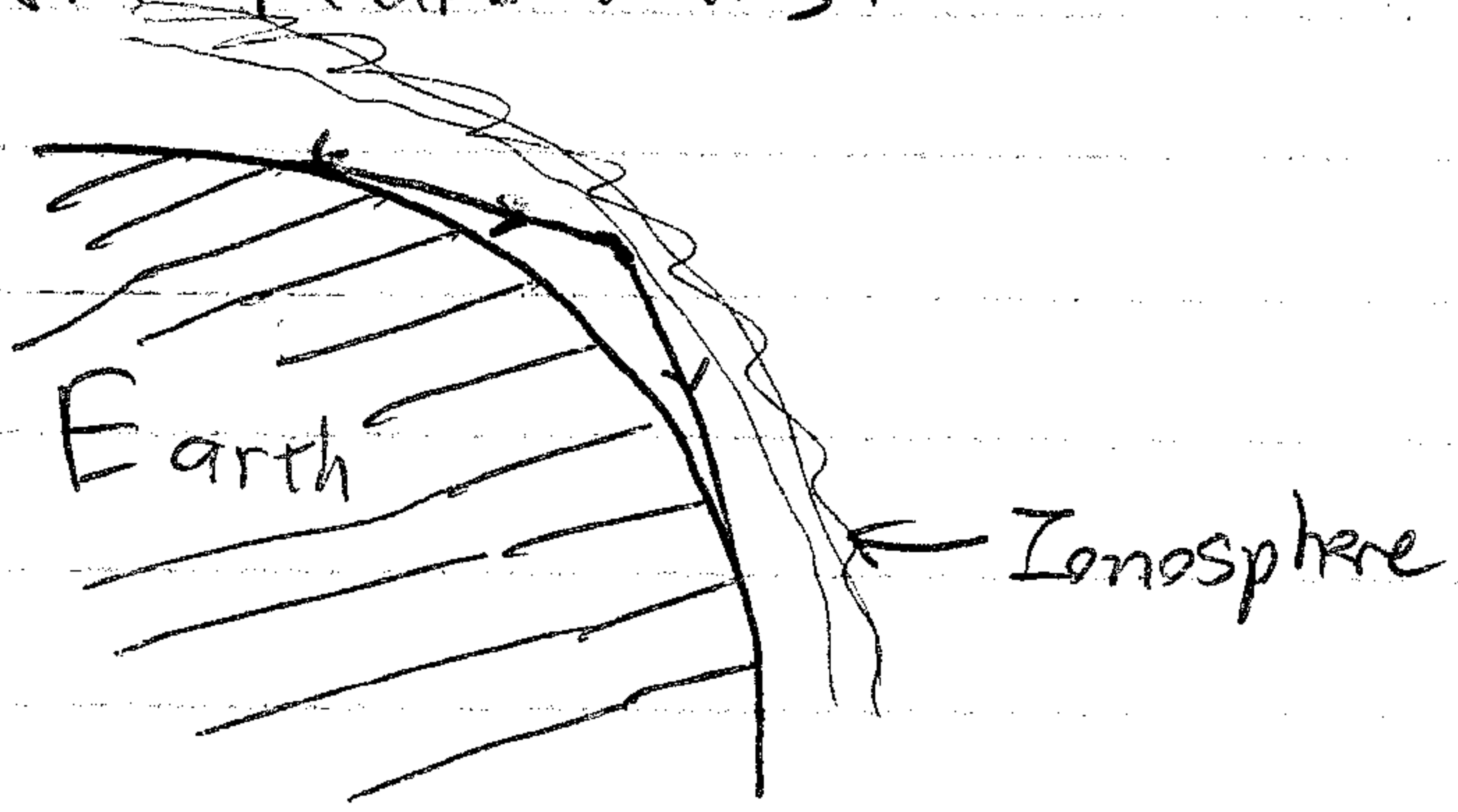
$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot (\underset{\sim}{V}_g \mathcal{E}) = 0}$$

where $\boxed{\mathcal{E} = \frac{\epsilon_0 \, \omega \, |\underset{\sim}{E}_{(0)}|^2}{2}}$ is analogous to wave energy.

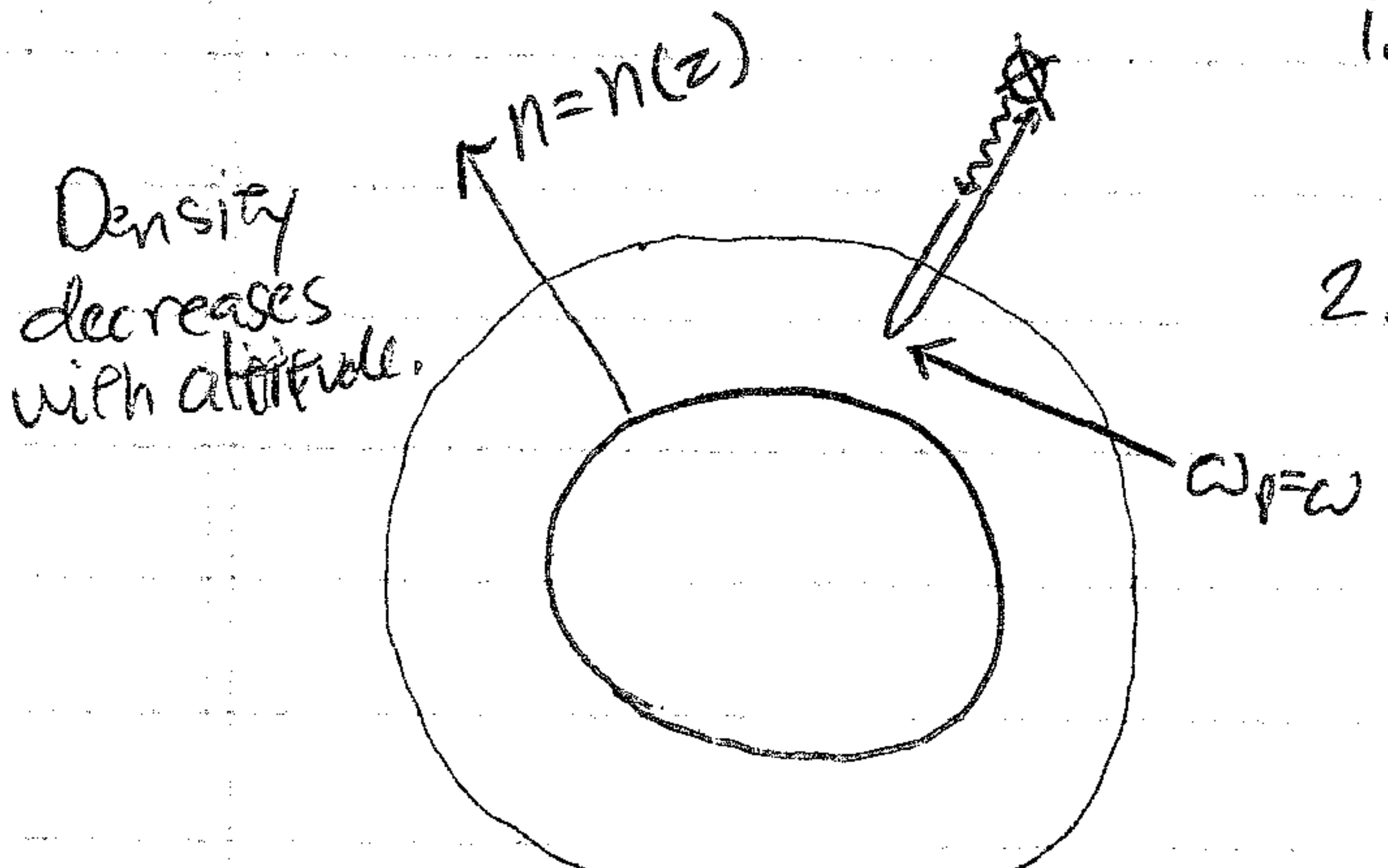
and $\underset{\sim}{V}_g = \left(\frac{\partial \omega}{\partial \underset{\sim}{k}}\right)_{x, t} = \frac{c^2 \underset{\sim}{k}}{\omega}$ in this case

IV. Applications:

A. AM Radio Waves:



B. MARSIS: Mars Advanced Radar for Subsurface and Ionosphere Sounding (on Mars Express spacecraft)



1. Radio wave at frequency ω is sent down into ionosphere
2. Radio wave reflects at $\omega = \omega_p$
3. By scanning frequency and measuring signal return time, you can get an altitude profile of density, $n(z)$