Lecture #6: Ray Tracing in Inhomogeneous Plasmas

I. Introduction

A. Inhomogeneous Plasmas

1. Everything we have learned about waves so far has been for homogeneous plasmas → we can Fourier analyze & solve.
2. In reality, completely homogeneous plasmas do not exist (but we'll see the lowest order properties of waves in inhomogeneous plasmas corresponds to the homogeneous solution).

3. Ray Tracing

Ray tracing is a technique used to solve for fields in many physical situations:

a. Radio waves in plasmas
b. Propagation of seismic waves in the earth & the Sun
c. General relativistic bending of light by gravity in galaxy clusters

B. Wave Propagation in an Inhomogeneous, Cold, Unmagnetized Plasma

1. For simplicity, we'll consider a cold, unmagnetized plasma with an equilibrium density gradient \( n_0 = n_0(x, t) \)
   a. Since \( \omega_p^2(x, t) = \frac{n_0(x, t)}{\epsilon_0} \left( \frac{me^2}{m_i} \right) \) (assuming \( n_0 i = n_0 e \))
   
   \( \Rightarrow \) The plasma frequency changes in space and time.

2. From Lee #22 of 29/194 (Eq. III.C.3.b.)
   a. \( c^2 \Delta x(k \times E_i) = \omega_p^2 E_i - \omega^2 E_i \) where \( \omega_p^2 = \omega_{p0}^2 + \omega_{p1}^2 \)
   b. We can go through all the same steps without Fourier transforming together:

\[
-c^2 \nabla \times (\nabla \times E_i) = \omega_p^2(x, t) E_i + \frac{\partial^2 E_i}{\partial t^2} \tag{1}
\]
The electron cyclotron wave. A Plasma.

**Physical Description**

An electromagnetic wave is created as the electrons move in the plasma. The wave is right-handed, meaning it is polarized in the direction of the electric field. The relationship between the wave and the plasma is given by

\[ \mathbf{E}_0 = \mathbf{E} \text{, } \mathbf{B}_0 = \mathbf{B} \]

The wave velocity is given by

\[ v = \frac{\mathbf{E}_0}{\mathbf{B}_0} \]

The wave equation for the electron-cyclotron wave is

\[ \frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial x^2} \]

The solution of the mode frequency is given by

\[ \omega^2 = \frac{\omega_0^2}{c^2} \left( 1 - \frac{x^2}{c^2} - \frac{y^2}{c^2} - \frac{z^2}{c^2} \right) \]

The electron-cyclotron wave is given by

\[ E_x = E_0 \cos k_x x \]

\[ E_y = E_0 \cos k_y y \]

\[ E_z = E_0 \cos k_z z \]

\[ B_x = B_0 \cos k_x x \]

\[ B_y = B_0 \cos k_y y \]

\[ B_z = B_0 \cos k_z z \]

Where

\[ \omega_0^2 = \frac{\sigma_0^2}{c^2} \]
4. Expand solution $E_1$ in powers of $\epsilon$: $E_1 = E_{1(0)} + \epsilon E_{1(1)} + \epsilon^2 E_{1(2)} + \cdots$

C. $O(\epsilon)$ Solution:

1. $k \times (k \times E_0) = \frac{c^2}{\epsilon^2} E_{1(0)}$

   a. This just gives the dispersion relation for a homogeneous plasma.

2. Focus on the Modified Light Wave $\Rightarrow$ take $k_0 E_{1(0)} = 0$

   $\Rightarrow \frac{\omega^2(x, t)}{c^2} - \omega_p^2(x, t) + k^2(x, t) c^2 = 0$

   a. Usually, $\omega = \omega(x, k, t)$, but since $k = k(x, t)$, we may write $\omega = \omega(x, t)$

3a. Assuming we know $n(x, t)$, then $\omega_p^2(x, t)$ is known.

   b. This leaves us with 4 unknowns $[\omega(x, t), k(x, t)]$ and one equation.

   c. But, $\omega$ & $k$ are related $\Rightarrow$ Both obtained from one function, $s(x, t)$.

4. a. Remember, by definition, $\frac{\partial k}{\partial x} = -\nabla \omega$

   b. But $\omega = \omega(x, k(x, t), t)$, so

   $\nabla \omega = \frac{\partial \omega}{\partial x} (\omega_{x1})_{,1} + \frac{\partial \omega}{\partial k} \left( \frac{\partial k}{\partial x} \right)_{,1} = \frac{\partial \omega}{\partial x} + \nabla k \times \left( \frac{\partial k}{\partial x} \right)_{,1}$

   c. Subtle point! $\nabla k = \nabla (\nabla s) \Rightarrow$ This is a symmetric tensor, so we may write $\nabla k \cdot \frac{\partial s}{\partial k} = \frac{\partial \omega}{\partial x} \cdot \nabla k$

   d. This gives: $\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial k} \cdot \nabla k = -\frac{\partial \omega}{\partial x}$

5. Remember, group velocity $v_g = \frac{\partial \omega}{\partial k}$ at a given point. $x$. 
6. Lagrangian Frame:
   a. Follow a point moving with group velocity: \( \frac{dx}{dt} = y_j = \left( \frac{\partial w}{\partial k} \right)_{x,t} \)
   b. The Lagrangian (or convective, substantial) derivative is:
      \[ \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \mathbf{\nabla} \]
   c. Thus:
      \[ \frac{d}{dt} = -\left( \frac{\partial w}{\partial x} \right) \]
      \[ \frac{d}{dt} = \left( \frac{\partial w}{\partial x} \right)_{x,t} \]

D. The Ray Equations

1. \[ \frac{d}{dt} = -\left( \frac{\partial w}{\partial x} \right)_{k,t} \]
   \[ \frac{dx}{dt} = \left( \frac{\partial w}{\partial k} \right)_{x,t} \]
   \[ \frac{dw}{dt} = \left( \frac{\partial w}{\partial k} \right)_{x,k} \]

The Ray Equations are completely analogous to Hamilton's equations under the change:

- \( \omega \rightarrow H \)
- \( x \rightarrow \mathbf{x} \)
- \( k \rightarrow \mathbf{p} \)

III. Solving the Ray Equations

A. i.

After some time:

- \( x, k, \omega \)
- End by integrating ray equations

⇒ trajectory
Lecture # 6 (Continued)

III. A. (Continued)

2. a. Solving for \( s(\xi, t) \): We can find \( s(\xi, t) \) by integrating along the ray:
\[
\frac{ds}{dt} = \frac{dx}{dt} + y_j \cdot \nabla s = -c_0(\xi, t) + y_j \cdot k(\xi, t)
\]
b. But this only gives \( s \) along the ray.
c. Using a computer, we can scan at a hemisphere of points.
d. By integrating along many such paths, we can eventually find \( s(\xi, t) \) over all space (by interpolation).

B. Amplitudes:
1. Our original solution assumed \( E_1(\xi, t) = E_1(\xi, t) e^{i s(\xi, t)} \)
2. We have solved for the eikonal \( s(\xi, t) \), but usually we want to know the amplitude as well.
3. To solve for amplitude, we go to the next order in the expansion.

2. \( O(1) \):
   a. \[
   \frac{k \cdot (k \times E(0)) - cu^2/c^2}{c^2} E(0) = \frac{i}{c} k \cdot (\nabla \times E_1(0)) + \nabla \times (k \times E_0(0)) - \frac{i}{c^2} E_0(0) - \frac{i}{c^2} E_1(0)
   \]
   We don’t need to know \( E(0) \) We want to find \( E_1(0) \)
   b. Annihilate \( E_1 \) by choosing solution with \( E_1^{*} \):
   \[
   E_0^{*} \left[ k \cdot (k \times E_0(0)) - \frac{cu^2/c^2}{c^2} E(0) \right] = (E_0^{*} \cdot k)(k \times E_0(0)) + \frac{k^2 - u^2/c^2}{c^2} E_0^{*} E_0(0) = 0
   \]
   c. We may then add the resulting RHS to its complex conjugate and manipulate.
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III. B. (Continued)

3. Continuity Equation for Wave Energy:

\[ \frac{\partial E}{\partial t} + \nabla \cdot (\nu E) = 0 \]

where

\[ E = \frac{\rho \omega |E_{\text{inc}}|^2}{2} \]

is analogous to wave energy.

and

\[ \nu = \frac{\partial (\omega E)}{\partial E} = \frac{c^2 k}{\omega} \]

in this case.

IV. Applications:

A. AM Radio Waves:

![Diagram of Earth and Ionosphere with radio waves]

B. MARSIS: Mars Advanced Radar for Subsurface and Ionosphere Sounding

(On Mars Express spacecraft)

1. Radio wave at frequency \( \omega \) is sent down into ionosphere

2. Radio wave reflects off \( \omega = \omega_p \)

3. By scanning frequency and measuring signal return time, you can get altitude profile of density, \( n(z) \)