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Lecture #8 Normal Mode Analysis and the Energy Principle Howes ①I. Properties of the Linear Force OperatorA. Review

1. Last time we derived the linear force operator for small displacements $\underline{\xi}_1$

$$\rho_0 \frac{\partial^2 \underline{\xi}_1}{\partial t^2} = \underline{F}(\underline{\xi}_1) \quad \text{where}$$

$$\underline{F}(\underline{\xi}_1) = \nabla[\underline{\xi}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{\xi}_1] + \frac{(\nabla \times \underline{B}_0) \times [\nabla \times (\underline{\xi}_1 \times \underline{B}_0)]}{\mu_0} + \frac{(\nabla \times [\nabla \times (\underline{\xi}_1 \times \underline{B}_0)]) \times \underline{B}_0}{\mu_0}$$

2. Also, recall the conserved energy in Ideal MHD:

$$E = \int d^3x \left[\frac{1}{2} \rho U^2 + \frac{p}{\gamma-1} + \frac{B_0^2}{2\mu_0} \right]$$

B. Expansion of MHD Energy in orders of $\underline{\xi}_1$

1. Just as we did for the simple mechanical system in I.B. of Lecture #7, we can split the MHD conserved energy into orders of $\underline{\xi}_1$.

$$\mathcal{O}(|\underline{\xi}_1|^0) \quad E_0 = \int d^3x \left[\frac{p_0}{\gamma-1} + \frac{B_0^2}{2\mu_0} \right]$$

$$\mathcal{O}(|\underline{\xi}_1|^1) \quad E_1 = \int d^3x \underline{\xi}_1 \cdot \left[\nabla p_0 - \frac{(\nabla \times \underline{B}_0) \times \underline{B}_0}{\mu_0} \right] \quad \left(\text{This } [] = 0 \text{ in MHD equilibrium} \right)$$

$$\mathcal{O}(|\underline{\xi}_1|^2) \quad E_2 = \int d^3x \left[\underbrace{\frac{1}{2} \rho_0 \left| \frac{\partial \underline{\xi}_1}{\partial t} \right|^2}_{\text{Kinetic Energy}} \right] + \underbrace{\delta W(\underline{\xi}_1, \underline{\xi}_1)}_{\text{Potential Energy}}$$

We'll derive form of δW soon.

C. Self-Adjoint Property

1. We can differentiate E_2 in time to determine a form of δW :

$$\frac{\partial E_2}{\partial t} = \int d^3x \frac{\partial}{\partial t} \left[\frac{1}{2} \rho_0 \left| \frac{\partial \underline{\xi}_1}{\partial t} \right|^2 \right] + \frac{\partial}{\partial t} \left[\delta W(\underline{\xi}_1, \underline{\xi}_1) \right]$$

a. NOTE: $\frac{\partial}{\partial t} \left[\frac{1}{2} \rho_0 \left| \frac{\partial \underline{\xi}_1}{\partial t} \right|^2 \right] = \rho_0 \frac{\partial \underline{\xi}_1}{\partial t} \cdot \frac{\partial^2 \underline{\xi}_1}{\partial t^2}$

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b. NOTE: $\frac{\partial}{\partial t} [\delta W(\underline{\xi}_1, \underline{\xi}_1)] = \delta W\left(\frac{\partial \underline{\xi}_1}{\partial t}, \underline{\xi}_1\right) + \delta W\left(\underline{\xi}_1, \frac{\partial \underline{\xi}_1}{\partial t}\right)$

2. But, Conservation of energy implies $\frac{\partial E_2}{\partial t} = 0$, so

$$\int d^3x \frac{\partial \underline{\xi}_1}{\partial t} \cdot \underline{F}(\underline{\xi}_1) = - \left[\delta W\left(\frac{\partial \underline{\xi}_1}{\partial t}, \underline{\xi}_1\right) + \delta W\left(\underline{\xi}_1, \frac{\partial \underline{\xi}_1}{\partial t}\right) \right]$$

where we have used $\rho_0 \frac{\partial^2 \underline{\xi}_1}{\partial t^2} = \underline{F}(\underline{\xi}_1)$

3. This statement must be true at $t=0$ when I can choose $\underline{\xi}_1$ and $\frac{\partial \underline{\xi}_1}{\partial t}$ arbitrarily as initial conditions.

\Rightarrow Therefore, it must be true for any chosen vectors $\underline{\xi}_1$ and $\underline{\eta}_1 = \left(\frac{\partial \underline{\xi}_1}{\partial t}\right)$

$$\int d^3x \underline{\eta}_1 \cdot \underline{F}(\underline{\xi}_1) = - \left[\delta W(\underline{\eta}_1, \underline{\xi}_1) + \delta W(\underline{\xi}_1, \underline{\eta}_1) \right]$$

4. This statement is clearly symmetric under exchange of $\underline{\xi}_1$ and $\underline{\eta}_1$, so

$$\boxed{\int d^3x \underline{\eta}_1 \cdot \underline{F}(\underline{\xi}_1) = \int d^3x \underline{\xi}_1 \cdot \underline{F}(\underline{\eta}_1)}$$

The Linear Force Operator \underline{F} is self-adjoint!

D. Form for $\delta W(\underline{\xi}_1, \underline{\xi}_1)$

1. The property above suggests the following form for δW :

$$\boxed{\delta W = -\frac{1}{2} \int d^3x \underline{\xi} \cdot \underline{F}(\underline{\xi})}$$

NOTE: From this point on, $\underline{\xi}$ is understood to be the linearized displacement, so I drop the subscript "1".

II. Normal Mode Analysis:

A. Representation as a Superposition of Normal Modes

1. An arbitrary mode has displacement $\xi_n(\underline{x}, t)$, which satisfies

$$\rho_0 \frac{\partial^2 \xi_n}{\partial t^2} = \underline{F}(\xi_n)$$

a. \underline{F} is a time-independent linear operator on $\xi_n(\underline{x}, t)$, so we can separate space & time dependence parts

$$\xi_n(\underline{x}, t) = \xi_n(\underline{x}) e^{-i\omega_n t}$$

where we assume a simple-harmonic form for time dependence.

b. Thus, we find

$$-\rho_0 \omega_n^2 \xi_n(\underline{x}) = \underline{F}[\xi_n(\underline{x})]$$

c. The general solution for an arbitrary $\xi(\underline{x}, t)$ is the sum of normal modes

$$\xi(\underline{x}, t) = \sum_n \xi_n(\underline{x}) e^{-i\omega_n t}$$

B. Properties of Normal Modes

1. Property I: ω_n^2 is always real.

Proof: a. Consider the complex conjugate of the equation of motion

$$-\rho_0 \omega_n^{2*} \xi_n^*(\underline{x}) = \underline{F}(\xi_n^*)$$

b. Dot with ξ_n and integrate over volume $\int d^3x$ By self-adjoint property.

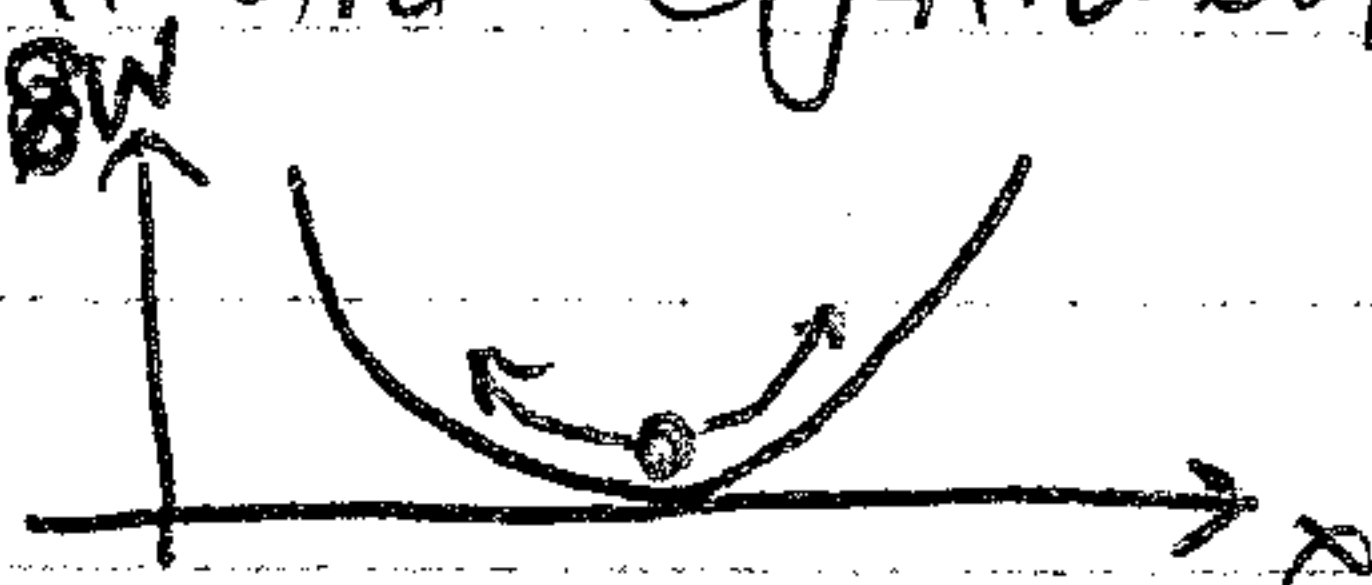
$$\begin{aligned} -\rho_0 \omega_n^{2*} \int d^3x |\xi_n|^2 &= \int d^3x \xi_n \cdot \underline{F}(\xi_n^*) = \int d^3x \xi_n^* \cdot \underline{F}(\xi_n) \\ &= -\rho_0 \omega_n^2 \int d^3x |\xi_n|^2 \end{aligned}$$

c. For $|\xi_n|^2$ non-zero, this leads to

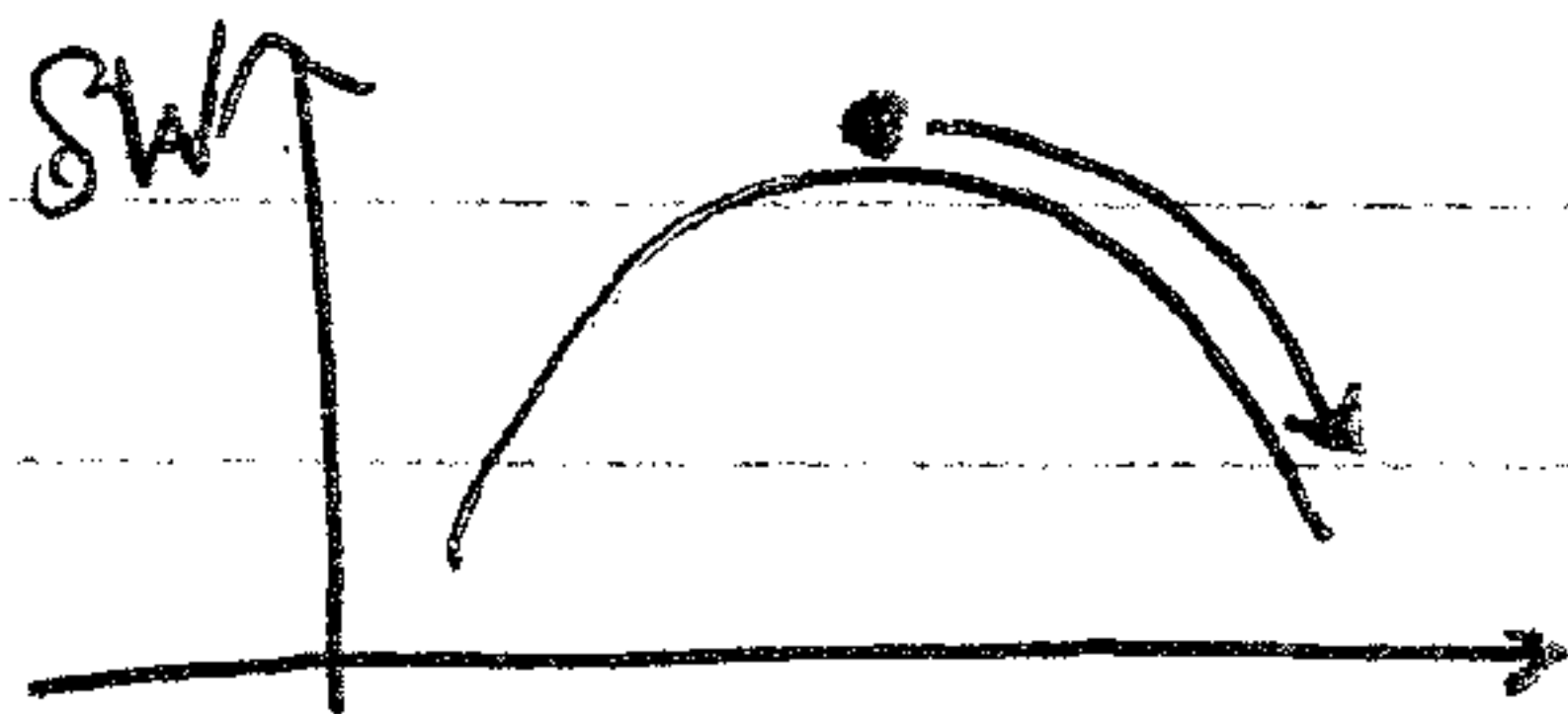
$$\boxed{\omega_n^{2*} = \omega_n^2} \Rightarrow \boxed{\omega_n^2 \text{ is always real}}$$

2. Implications of Property I:

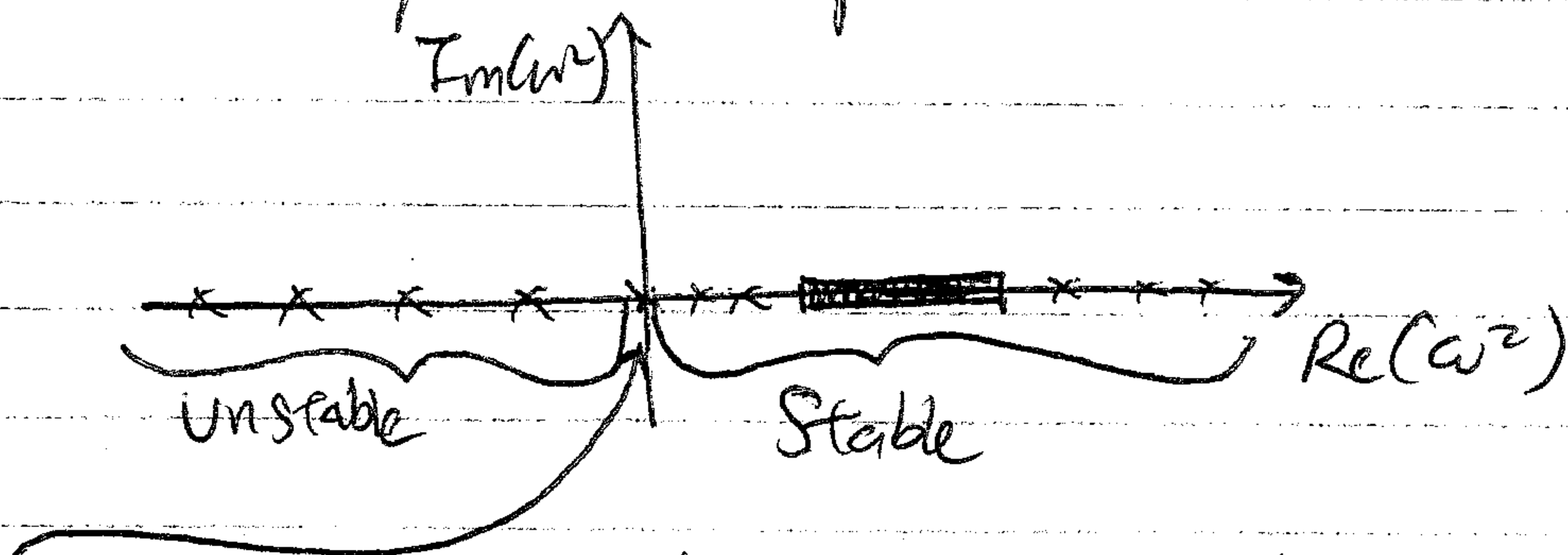
a. For $\omega_n^2 > 0$, ω_n is purely real and eigenfunction is oscillatory. \Rightarrow STABLE



b. For $\omega_n^2 < 0$, ω_n is purely imaginary and eigenfunction undergoes exponential growth due to one root. \Rightarrow UNSTABLE



c. Numerical Simplification: In solving for modes of the equation of motion (i.e. finding the frequency of the normal mode), one need only look for real ω^2 and need not search all of complex ω^2 space.



d. $\omega_n^2 = 0$ is the point of marginal stability separating stable from unstable solutions.

3. Property II: The eigenmodes of F are orthonormal,

$$\int d^3x \rho \sum_m^* \cdot \sum_n = \delta_{mn}.$$

(See Gurnee & Bhattacharjee for proof).
 sec 6.7.4

II. (Continued)

C. General Procedure for Solution by Normal Mode Analysis

- The procedure is analogous to the solution of the linear dispersion relation for a given system.
- Of course, we take not homogeneous ~~conditions~~ but use the equilibrium solutions for $\rho_0(x)$, $B_0(x)$
- Also, unlike MHD waves in a homogeneous plasma, sources of free energy are present so we may find many unstable eigenmodes (with ω_n purely imaginary).
- In fact, stability is more often the exception than the rule.

2. a. Begin with $\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \underline{F}(\xi) \Rightarrow -\rho_0 \omega_n^2 \underline{\xi}_n = \underline{F}(\underline{\xi}_n)$

- For many systems, we can simplify $\underline{F}(\underline{\xi})$ because some of the terms are zero.
- ~~The vector equation~~ Symmetries of the system can be used to reduce at least some components of the vector operators in $\underline{F}(\underline{\xi})$ to algebraic operators (for periodicity, we can use a Fourier decomposition)
- The vector equation yields a 3×3 matrix equation which can be solved for the eigenfrequencies ω_n .

- This method yields ^(Stable) frequencies or unstable growth rates for each mode, and can be used to reconstruct the eigenfunctions.
 - The somewhat complicated normal mode analysis often gives us more information than we need.
 - Often, we care only if a system is unstable.
- \Rightarrow The Energy Principle is a more easily applied, yet extremely powerful, technique that determines stability.

III. The Energy Principle

A. Necessary and Sufficient Conditions for Stability

1. Instability is relatively easy to prove:

a. Choose a physically motivated perturbation ξ

b. Show that this perturbation leads to $\delta W < 0$.

2. Stability is much more difficult to prove

a. Must show that no perturbation can lead to $\delta W < 0$

b. Using the energy principle, one may minimize δW with respect to all possible perturbations

c. If δW_{\min} is positive, the system is stable.

3. Remember $E_2 = \underbrace{\int d^3x \left[\frac{1}{2} \rho_0 \left(\frac{\partial \xi}{\partial t} \right)^2 \right]}_{=\delta K} + \delta W(\xi, \xi) = \underline{\text{constant}}$

So $E_2 = \delta K + \delta W$. Also, by definition, $\delta K \geq 0$

4. Theorem I: If $\delta W \geq 0$ for all ξ then the system is stable.

$\Rightarrow \delta W \geq 0$ for all ξ is sufficient for stability

Proof: a. If $\delta W \geq 0$,

$$0 \leq \delta K = E_2 - \delta W \leq E_2$$

b. Thus δK is bounded from above, No unbounded growth of kinetic energy is possible so plasma is stable. QED.

5. Theorem II: If for some function ξ , $\delta W(\xi, \xi) < 0$, then the system is unstable.

$\Rightarrow \delta W \geq 0$ for all ξ is also necessary for stability.

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 III, A. 5. (Continued)

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Proof: a. Consider a displacement initially such that $\xi(x, 0) \neq 0$ but $\frac{\partial \xi}{\partial t}(x, 0) = 0$ (displaced but at rest).

b. Let this ξ lead to $\delta W < 0$.

c. At time $t=0$, $E_2 = \delta K + \delta W < 0 \Rightarrow E_2 < 0$.

d. Define $I(t) = \frac{1}{2} \int d^3x \rho_0 |\xi|^2$

e. Then
$$\begin{aligned} \frac{d^2 I}{dt^2} &= \frac{1}{2} \int d^3x \rho_0 \left[2 \left| \frac{\partial \xi}{\partial t} \right|^2 + \xi^* \cdot \frac{\partial^2 \xi}{\partial t^2} + \xi \cdot \frac{\partial^2 \xi^*}{\partial t^2} \right] \\ &= \frac{1}{2} \int d^3x \left[2 \rho_0 \left| \frac{\partial \xi}{\partial t} \right|^2 + \underbrace{\xi^* \cdot F(\xi) + \xi \cdot F(\xi^*)}_{= 2 \xi^* \cdot F(\xi) = -4 \delta W} \right] \end{aligned}$$

f. Thus $\frac{d^2 I}{dt^2} = 2(\delta K - \delta W)$.

but $E_2 = \delta K + \delta W$ so $\delta W = E_2 - \delta K \Rightarrow \frac{d^2 I}{dt^2} = 2(2\delta K - E_2)$

g. $\delta K \geq 0$, so let's take $\delta K = 0$. Then $\frac{d^2 I}{dt^2} = -2E_2 > 0$
 because $E_2 < 0$.

h. Thus I increases without bound if $\delta W < 0$. \Rightarrow UNSTABLE. QED.

~~Proof: a. Consider a displacement initially such that $\xi(x, 0) \neq 0$ but $\frac{\partial \xi}{\partial t}(x, 0) = 0$ (displaced but at rest).~~

6. Thus $\delta W \geq 0$ for all ξ is a necessary and sufficient condition for stability.

B. Forms for δW :

1. We can use $\delta W = \frac{1}{2} \int d^3x \xi \cdot F(\xi)$ and linear force operator $F(\xi)$ and use it forms for δW .

III. B. (Continued)

2. After substantial algebra (see Gurnett & Bhattacharjee Sec 6.7.6) we arrive at the form:

$$\delta W = \frac{1}{2} \int d^3x \left[\underbrace{\frac{|\nabla \times (\underline{\xi} \times \underline{B}_0)|^2}{\mu_0}}_{\text{Magnetic Tension and compression}} + \underbrace{\gamma p_0 |\nabla \cdot \underline{\xi}|^2}_{\text{Thermal Compression}} - \underbrace{\underline{\xi}^* \cdot \underline{j}_0 \times [\nabla \times (\underline{\xi} \times \underline{B}_0)]}_{\text{"Kink" Drive}} - \underbrace{\underline{\xi}^* \cdot \nabla (\underline{\xi} \cdot \nabla p_0)}_{\text{"Interchange" or "Ballooning" Drive}} \right]$$

positive \Rightarrow Stabilizing
potentially destabilizing

3a. Since thermal compression term is always stabilizing, taking incompressible motions ($\gamma \rightarrow \infty$) are always more stable than compressible motions.

b. A ~~press~~ fluid with pressure independent of volume ($\gamma \rightarrow 0$) is the most unstable.

4. A more complete treatment of a finite volume plasma confined by vacuum magnetic fields includes surface terms and vacuum field energy terms in δW .

C. Application of Energy Principle to Evaluate Stability

1. One may calculate δW for a given equilibrium $p_0(x)$ and $B_0(x)$ for an arbitrary $\underline{\xi}$

2. Eventually, one can minimize δW with respect to each component of $\underline{\xi}$. For example, take $\frac{\partial \delta W}{\partial \xi_x} = 0$ to

find the minimum δW at the solutions for $\underline{\xi}_x$ minima.

3. Ultimately, one reaches a final value δW_{\min} .

If $\delta W_{\min} < 0$, unstable. Otherwise stable.