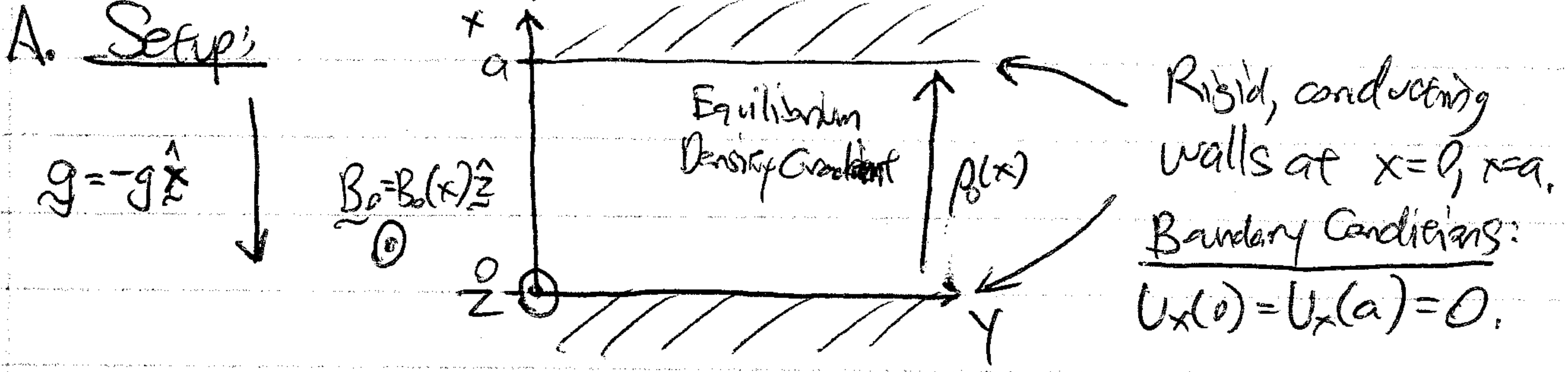
Lecture #9: MHD Stability Analysis of Rayleigh–Taylor Instability

I. Normal Mode Analysis

A. Setup:



1. Density has exponential form in direction of gravity ($\hat{x}$)

a. $\rho_0(x) = \rho_{00}\, e^{-\frac{x}{H}}$ $\quad H \equiv$ Scale height of density

b. For $H>0$, density decreases with height (stable)
$H<0$, density increases with height (unstable)

2. a. For simplicity, we assume $\frac{\partial}{\partial z}=0$ (No variation along mean field).
NOTE: Such variations would bend the magnetic field lines, leading to magnetic tension (which stabilizes instability)

b. We also assume incompressible motion, $\nabla\cdot\mathbf{U}_1=0$

3. We want to analyze this problem ~~for~~ stability.
a. Normal Mode Analysis
b. Energy Principle

B. Using Linear Force Operator

1. We could use
$$-\rho_0\omega^2\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$$
to solve for the characteristic frequencies. But instead, we'll begin from equation of motion.

Lecture #9 (Continued)

I.C. Using Equation of Motion:

1. Momentum Eq:

$$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0} + \underbrace{\rho g}_{\text{gravity, where}}$$

$g = -g \hat{\underset{\sim}{x}}$

2. Linearize about Equilibrium:

$$\rho = \rho_0(x) + \epsilon \rho_1(\underset{\sim}{x})$$
$$\underset{\sim}{U} = \cancel{U_0(x)}^{0} + \epsilon \underset{\sim}{U}_1(\underset{\sim}{x})$$
$$\underset{\sim}{B} = B_0(x)\hat{\underset{\sim}{z}} + \epsilon \underset{\sim}{B}_1(\underset{\sim}{x})$$
$$p = p_0(x) + \epsilon p_1(\underset{\sim}{x})$$

vertical direction (scalar) — vector

3. Lowest Order: $\mathcal{O}(1)$: $0 = -\nabla\left(p_0 + \frac{B_0^2}{2\mu_0}\right) + \rho_0 g$

a. Equilibrium satisfies: $\boxed{\frac{\partial}{\partial x}\left(p_0 + \frac{B_0^2}{2\mu_0}\right) = -\rho_0 g}$ Static Equilibrium

b. Notation: $p_0' = \frac{\partial p_0}{\partial x}$, $B_0' = \frac{\partial B_0}{\partial x}$, $\rho_0' = \frac{\partial \rho_0}{\partial x}$

$$\Rightarrow \boxed{p_0' + \frac{B_0 B_0'}{\mu_0} = -\rho_0 g}$$

4. Next Order: $\mathcal{O}(\epsilon)$:

$$\underset{(1)}{\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}} = \underset{(2)}{-\nabla\left(p_1 + \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{\mu_0}\right)} + \underset{(3)}{\frac{(\underset{\sim}{B}_0 \cdot \nabla)\underset{\sim}{B}_1}{\mu_0}} + \underset{(4)}{\frac{(\underset{\sim}{B}_1 \cdot \nabla)\underset{\sim}{B}_0}{\mu_0}} + \underset{(5)}{\rho_1 g}$$

a. Term (3) $(\underset{\sim}{B}_0 \cdot \nabla)\underset{\sim}{B}_1 = B_0 \cancel{\frac{\partial}{\partial z}}^{0} \underset{\sim}{B}_1 = 0$

b. Term (4) $(\underset{\sim}{B}_1 \cdot \nabla)\underset{\sim}{B}_0 = B_x \frac{\partial B_0}{\partial x}\hat{\underset{\sim}{z}} = B_0' B_x \hat{\underset{\sim}{z}}$

c. We can eliminate Term (2) by taking the curl of this equation:

$(\nabla \times \nabla \phi = 0)$. — ϕ: scalar function.

Lecture #9 (Continued)

I.C. 4. (Continued)

d. $\nabla\times\left(\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}\right) = \nabla\times\left(B_0' B_x \hat{\underset{\sim}{z}}\right) + \nabla\times\left(-\rho_1 g \hat{x}\right)$

e. NOTE: Since $\frac{\partial}{\partial z}=0$, $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$

f. Let's find the $\hat{\underset{\sim}{z}}$- component:

1. $\hat{\underset{\sim}{z}}\cdot\left[\nabla\times\left(\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}\right)\right] = \frac{\partial}{\partial x}\left[\rho_0 \frac{\partial U_y}{\partial t}\right] - \frac{\partial}{\partial y}\left[\rho_0 \frac{\partial U_x}{\partial t}\right]$

$$= \rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t}$$

2. $\hat{\underset{\sim}{z}}\cdot\left[\nabla\times\left(B_0' B_x \hat{\underset{\sim}{z}}\right)\right] \equiv 0$

3. $\hat{\underset{\sim}{z}}\cdot\left[\nabla\times\left(-\rho_1 g \hat{\underset{\sim}{x}}\right)\right] = -\frac{\partial}{\partial y}\left[-\rho_1 g\right] = g\frac{\partial \rho_1}{\partial y}$

g. Thus, we find $\boxed{\rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t} = g\frac{\partial \rho_1}{\partial y}}$

5. Fourier Transform in y and t: $\underset{\sim}{U}_1(\underset{\sim}{x}) = \underset{\sim}{U}_1(x)\, e^{i(k_y y - \omega t)}$

a. Thus $\frac{\partial}{\partial t} = -i\omega$ $\quad \frac{\partial}{\partial y} = ik_y$

b. This yields $\boxed{\rho_0' U_y + \rho_0 \frac{\partial U_y}{\partial x} - \rho_0 i k_y U_x = -\frac{k_y g}{\omega}\rho_1}$

6. We assume incompressible motion $\nabla\cdot\underset{\sim}{U}_1 = 0$

$\boxed{\frac{\partial U_x}{\partial x} + i k_y U_y = 0}$ $\quad\Rightarrow U_y = \frac{i}{k_y}\frac{\partial U_x}{\partial x}$

7. Continuity Equation $\quad \frac{\partial \rho}{\partial t} + \underset{\sim}{U}\cdot\nabla\rho + \rho\nabla\cdot U = 0$

a. $\mathcal{O}(\epsilon)$: $\frac{\partial \rho_1}{\partial t} + \underset{\sim}{U}_1\cdot\nabla\rho_0 + \cancel{\rho_0\nabla\cdot\underset{\sim}{U}_1}^{\,0} = 0$

b. $\boxed{-i\omega\rho_1 + U_x\rho_0' = 0}$ $\quad\Rightarrow\quad \rho_1 = \frac{-i}{\omega}U_x\rho_0'$

I.C. (Continued)

8. Eliminate U_y & ρ_1 in favor of U_x:

a. This yields:
$$\frac{\partial^2 U_x}{\partial x^2} + \frac{\rho_0'}{\rho_0}\frac{\partial U_x}{\partial x} - k_y^2\left[1+\frac{g}{\omega^2}\left(\frac{\rho_0'}{\rho_0}\right)\right]U_x = 0$$

9. Note: Since $\rho_0(x) = \rho_{00} e^{-\frac{x}{H}}$, we have $\frac{\rho_0'}{\rho_0} = -\frac{1}{H}$
$$\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H}\frac{\partial U_x}{\partial x} - k_y^2\left(1-\frac{g}{\omega^2 H}\right)U_x = 0$$

10. We can solve this with the help of an integrating factor.

a. Let $U_x(x) = f(x)\, e^{+\frac{x}{2H}}$

b. This gives $\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H}\frac{\partial U_x}{\partial x} = \left(\frac{\partial^2 f}{\partial x^2} - \frac{f}{4H^2}\right)e^{+\frac{x}{2H}}$

c. Thus, we find:
$$\frac{\partial^2 f}{\partial x^2} + \alpha^2 f = 0 \quad \text{where} \quad \alpha^2 = k_y^2\left(\frac{g}{H\omega^2}-1\right) - \frac{1}{4H^2}$$

11. The function $f(x)$ must satisfy the boundary conditions
$$U_x(0) = U_x(a) = 0 \quad \Rightarrow f(0) = f(a) = 0$$

a. General Solution: $f(x) = f_0 \sin\alpha x + f_1 \cos\alpha x$

b. $f(0) = f_0(0) + f_1(1) = 0 \quad \Rightarrow f_1 = 0.$

c. $f(a) = f_0 \sin(\alpha a) = 0 \quad \Rightarrow \alpha = \frac{n\pi}{a}$ for $n = 1, 2, 3, \ldots$

d. We therefore have eigenfunctions f_n with mode number n.

12. Solve for frequency: $\frac{n^2\pi^2}{a^2} = k_y^2\left(\frac{g}{H\omega_n^2} - 1\right) - \frac{1}{4H^2}$

a. $\frac{n^2\pi^2}{a^2} + k_y^2 + \frac{1}{4H^2} = \frac{g}{H}\frac{k_y^2}{\omega_n^2}$

Lecture #9 (Continued)

I. C.R. (Continued)

b. $$\boxed{\omega_n^2 = \left(\frac{g}{H}\right) \frac{4H^2 k_y^2 a^2}{\underbrace{a^2 + 4H^2(n^2\pi^2 + k_y^2 a^2)}_{\text{positive definite}}}}$$

c. For $H > 0$, $\omega_n^2 > 0 \Rightarrow$ Oscillating function $\Rightarrow$ STABLE
For $H < 0$, $\omega_n^2 < 0 \Rightarrow$ UNSTABLE.

d.1. Max growth rate $\omega_n^2 = \frac{g}{H}$ as $k_y \to \infty$
2. Growth rate $\to 0$ as $k_y \to 0$
3. Lower vertical mode numbers n have faster growth.

II. Energy Principle:

A. Gravitational Force Term:

1. In Linear Force Operator $\underset{\sim}{F}(\underset{\sim}{\xi})$, we must add gravity term:
$+\rho_1 g$

a. From Lecture #7, II.A.4.b.2, we have $\rho_1 = -\underset{\sim}{\xi}\cdot\nabla\rho_0 - \rho_0 \nabla\cdot\underset{\sim}{\xi} = 0$

b. For incompressible motion, $\nabla\cdot\underset{\sim}{\xi} = 0$, so $\rho_1 = -\underset{\sim}{\xi}\cdot\nabla\rho_0 = -\xi_x \rho_0'$

c. Thus $+\rho_1 \underset{\sim}{g} = (-\xi_x \rho_0')(-g\underset{\sim}{\hat{x}}) = \rho_0' g \xi_x \underset{\sim}{\hat{x}}$

2. This gives:

$$\underset{\sim}{F}(\underset{\sim}{\xi}) = \nabla\left[\underset{\sim}{\xi}\cdot\nabla p_0 + \gamma p_0 \cancelto{0}{\nabla\cdot\underset{\sim}{\xi}}\right] + \frac{(\nabla\times\underset{\sim}{B}_0)\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0)] + (\nabla\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0)])\times\underset{\sim}{B}_0}{\mu_0}$$
$$+ \rho_0' g \xi_x \underset{\sim}{\hat{x}}$$

3. For the energy principle, we must add this term:

$$-\frac{1}{2}|\xi_x|^2 g \rho_0'$$

II. (Continued)

B. Using Energy Principle

1. With the added gravitational potential term, we have

$$\delta W = \frac{1}{2}\int d^3x \left\{ \frac{|\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B_0})|^2}{\mu_0} + \gamma p_0 |\cancelto{0}{\nabla\cdot\underset{\sim}{\xi}}|^2 - \underset{\sim}{\xi}^*\cdot \underset{\sim}{j_0}\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B_0})] \right.$$
① ② ③
$$\left. - \underset{\sim}{\xi}^*\cdot\nabla(\underset{\sim}{\xi}\cdot\nabla p_0) - |\xi_z|^2 g\rho_0' \right\}$$
④ ⑤

2. a. $\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B_0}) \overset{\text{NRL (10) p.4}}{=} \underset{\sim}{\xi}(\cancelto{0}{\nabla\cdot\underset{\sim}{B_0}}) - \underset{\sim}{B_0}(\cancelto{0}{\nabla\cdot\underset{\sim}{\xi}}) + \underbrace{(\underset{\sim}{B_0}\cdot\nabla)\underset{\sim}{\xi}}_{\cancelto{0}{B_0\partial_z}} - (\underset{\sim}{\xi}\cdot\nabla)\underset{\sim}{B_0}$

$= -B_0'\xi_x\hat{\underset{\sim}{z}}$

b. $\underset{\sim}{j_0} = \frac{\nabla\times\underset{\sim}{B_0}}{\mu_0} = \frac{1}{\mu_0}\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right)\times(B_0(x)\hat{z}) = \frac{-1}{\mu_0}B_0'\hat{y}$

3. TERM ①: $= (B_0')^2|\xi_x|^2$

4. TERM ③: $= -\underset{\sim}{\xi}^*\cdot\left(-\frac{B_0'}{\mu_0}\hat{y}\right)\times\left(-B_0'\xi_x\hat{\underset{\sim}{z}}\right) = -\underset{\sim}{\xi}^*\cdot\left(\frac{(B_0')^2}{\mu_0}\xi_x\hat{x}\right) = -\frac{(B_0')^2}{\mu_0}|\xi_x|^2$

a. Thus, Term ① + Term ③ = 0 ✓

5. Term ④: $-\underset{\sim}{\xi}^*\cdot\nabla\underbrace{[(\underset{\sim}{\xi}\cdot\nabla)p_0]}_{\text{Scalar} = f}$

a. NOTE: NRL (7) p.4 $\nabla\cdot(f\underset{\sim}{A}) = f\nabla\cdot\underset{\sim}{A} + \underset{\sim}{A}\cdot\nabla f$

$\nabla\cdot(\underset{\sim}{\xi}^* f) = \cancelto{0}{f\nabla\cdot\underset{\sim}{\xi}^*} + \underset{\sim}{\xi}^*\cdot\nabla f$

b. Thus $\int d^3\underset{\sim}{x}\left\{-\underset{\sim}{\xi}^*\cdot\nabla[(\underset{\sim}{\xi}\cdot\nabla)p_0]\right\} = -\int d^3\underset{\sim}{x}\,\nabla\cdot[\underset{\sim}{\xi}^*(\underset{\sim}{\xi}\cdot\nabla)p_0]$

By Divergence Thm $\overset{}{\underset{\text{NRL (28) p.6}}{=}} \oint d\underset{\sim}{S}\cdot[\underset{\sim}{\xi}^*(\underset{\sim}{\xi}\cdot\nabla)p_0] = \oint d\underset{\sim}{S}\cdot[\underset{\sim}{\xi}^*\xi_x p_0'] = 0$

By B.C.'s ξ_x at boundary $x=0,a$ is zero!
Periodic in y & z sums to zero.

6. Term ⑤: Only term left:

$$\boxed{\delta W = \frac{-1}{2}\int d^3\underset{\sim}{x}\,|\xi_z|^2 g\rho_0'}$$

II. B. (Continued)

7. Thus, for $\rho_0' > 0$ (density increasing with height),

$$\delta W < 0 \Rightarrow \text{UNSTABLE!}$$