29:293 Homework #2

Reading:	Required:	Read GB Chapter 8, Sections 8.1–8.3 (p.281–311)
	Optional:	Read BS Chapter 7, Sections 7.1–7.3 (p.252–268)

Due at the beginning of class, Thursday, February 14, 2013.

1. A plasma has a "spherical shell" distribution function given by

$$f_0(\mathbf{v}) = \frac{n_0}{4\pi C^2}\delta(|\mathbf{v}| - C)$$

where C is a constant.

   (a) Using the Fourier analysis approach, show that the dispersion relation for electrostatic waves in this plasma is $\omega^2 = \omega_p^2 + k^2C^2$.

   (b) What is the region of validity of this dispersion relation?

2. Show that the Laplace transform of $f(t) = \cosh(at)$ is given by

$$\tilde{f}(p) = \frac{p}{p^2 - a^2}.$$

3. Use the Residue Theorem to evaluate the inverse Laplace transform of

$$\tilde{f}(p) = \frac{1}{p^2 - a^2}.$$

4. Solution of Navier-Stokes Equations:
The Navier-Stokes Equations for the viscous evolution of a hydrodynamic fluid are given by:

$$\frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{U}$$

$$\rho\left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U}\right) = -\nabla p + \nu \rho \nabla^2 \mathbf{U}$$

$$\frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{U}$$

where ν is the coefficient of kinematic viscosity. Assume a wave vector of the form $\mathbf{k} = k\hat{\mathbf{z}}$. The initial conditions for this system at $t = 0$ are $\mathbf{U}(\mathbf{x}, 0) = \overline{U}\cos(kz)\hat{\mathbf{z}}$ and $\mathbf{U}'(\mathbf{x}, 0) = -\overline{U}\omega_0 \sin(kz)\hat{\mathbf{z}}$.
HINT: This is similar to a linear dispersion relation problem, so your first step is to linearize the Navier-Stokes equations.

   (a) Use the Laplace-Fourier transform method (Fourier transform in space, Laplace transform in time) to solve for the velocity $\mathbf{U}(\mathbf{x}, t)$ in complex terms. Be sure to use the definition $c_s^2 = \gamma p_0/\rho_0$ to simplify your calculations. HINT: The z-component of the velocity U_z is the non-trivial part of the solution.

   (b) Determine the evolution of the magnitude of the velocity $|\mathbf{U}(\mathbf{x}, t)|$.

   (c) In the weak damping limit $\nu^2k^2 \ll 4c_s^2$, what are the effective real frequency of oscillation (include the small, first order correction) and damping rate?

   (d) Qualitatively sketch the solution $\mathbf{U}(z = 0, t)$ in the case that $\nu^2k^2 < 4c_s^2$.

   (e) Qualitatively sketch $\mathbf{U}(z = 0, t)$ for the cases $\nu^2k^2 = 4c_s^2$. and $\nu^2k^2 > 4c_s^2$ on the same plot (but a different plot from part d).