29:293 Homework #6

Due at the beginning of class, Thursday, April 4, 2013.

1. Here we will apply a simplified version of Multiple-Timescale Analysis to the problem of particle motion in constant, uniform $\mathbf{E}$ and $\mathbf{B}$ fields.

   We assume a right-handed, orthonormal basis aligned with the direction of the magnetic field $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}})$ such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. The Lorentz Force Law is

   $$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

   for an electric field $\mathbf{E} = E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2 + E_\parallel\hat{\mathbf{b}}$ and a magnetic field $\mathbf{B} = B_0\hat{\mathbf{b}}$. For this problem, we will take the case $E_\parallel = 0$.

   (a) First, let us convert the dimensional form of the Lorentz Force Law above to a dimensionless equation. Derive the dimensionless form

   $$\frac{d\mathbf{v}'}{dt'} = \mathbf{E}' + \mathbf{v}' \times \hat{\mathbf{b}} \tag{1}$$

   for dimensionless quantities $t' = \omega_c t$, $\mathbf{v}' = \mathbf{v}/v_\perp$, and $\mathbf{E}' = \frac{\mathbf{E}}{B_0 v_\perp}$ where $v_\perp = \sqrt{v_1^2 + v_2^2}$.

   (b) Verify that the quantity $E' = |\mathbf{E}'|$ is dimensionless (in the SI system of units).

   (c) Show that the condition $E' \ll 1$ means that the $\mathbf{E} \times \mathbf{B}$ drift is slow compared to the perpendicular velocity, $|\mathbf{v}_E| \ll v_\perp$.

   (d) Assuming $E' \ll 1$, the timescales of the Larmor motion and the $\mathbf{E} \times \mathbf{B}$ drift are well separated. For the expansion parameter, take $\epsilon = E' \ll 1$. As an aid in the bookkeeping for the order of magnitude of each term, we can add an ϵ to the electric field term in our equation to remind us of its order,

   $$\frac{d\mathbf{v}'}{dt'} = \epsilon\mathbf{E}' + \mathbf{v}' \times \hat{\mathbf{b}} \tag{2}$$

   We'll assume a fast timescale t' and a slow timescale $\tau' = \epsilon t'$. Decompose the total velocity into rapidly varying piece $\mathbf{v}_1'$ and a smaller slowly varying piece $\mathbf{v}_2'$, $\mathbf{v}' = \mathbf{v}_1'(t') + \epsilon\mathbf{v}_2'(\tau')$.
   Write down the expansion of d/dt' assuming two timescales.

   (e) Derive the equation at $\mathcal{O}(1)$ and solve for $\mathbf{v}_1'(t')$ given the (dimensional) initial conditions at $t = 0$ of $\mathbf{v} = v_\perp\hat{\mathbf{e}}_1 + v_{\parallel 0}\hat{\mathbf{b}}$.

   (f) Derive the equation at $\mathcal{O}(\epsilon)$. Solve for $\mathbf{v}_2'(\tau')$. HINT: Do not forget to treat t' and τ' as independent variables.

   (g) Sum the solution for each order to get the total solution $\mathbf{v}'(t', \tau')$. Convert back to dimensional form to yield the final, complete solution $\mathbf{v}(t)$.

2. An electron of charge $q_e = -e$ and mass m_e and an proton of charge $q_e = e$ and mass $m_i = m_p$ are initially at rest at $\mathbf{x} = (0, 0, 0)$ in a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. An electric field is then turned on at $t = 0$ and increased linearly until time $t_1 = \frac{20\pi m_i}{eB_0}$, at which point the electric field is held constant,

   $$\mathbf{E}(t) = \begin{cases} 0 & t < 0 \\ E_0(t/t_1)\hat{\mathbf{y}} & 0 \le t \le t_1 \\ E_0\hat{\mathbf{y}} & t > t_1 \end{cases}$$

   Find the total current density as a function of time $\mathbf{j}(t)$ due to the drifts of the two particles (neglect the current due to the fast Larmor oscillation).

3. Laser Trapping: A charged particle in an unmagnetized plasma can be trapped by a spatially varying intense laser field. Using intereference of several lasers, the electric field near a charged particle is given by

$$\mathbf{E}(\mathbf{x}, t) = E_0[1 + (x/x_0)^2] \sin(\omega t - k_y y)\hat{\mathbf{x}}.$$

Calculate the velocity of the oscillation center U as a function of position x for a particle initially at rest at $t = 0$ at position $\mathbf{x} = (x_0, 0, 0)$. You may assume that the particle velocity v and laser frequency ω satisfy $v \ll \omega/k_y$ and $v/x_0 \ll \omega$.