

Lecture #12 Quasi-Linear Theory, Part II

Hawes ①

I. Review:

A. Last time, we calculated the Quasilinear Equations

$$1. \left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} \right) f_{s1} = \frac{q_s}{m_s} \frac{\partial \phi_1}{\partial z} \frac{\partial \langle f_s \rangle}{\partial v_z} \quad \left. \vphantom{\frac{\partial}{\partial t}} \right\} \leftarrow \text{Fast evolution of Fluctuations}$$

$$2. \frac{\partial^2 \phi_1}{\partial z^2} = - \sum_s \frac{q_s}{\epsilon_0} \int_{-\infty}^{\infty} dv_z f_{s1}$$

$$3. \frac{\partial \langle f_s \rangle}{\partial t} = \frac{q_s}{m_s} \frac{\partial}{\partial v_z} \left\langle f_{s1} \frac{\partial \phi_1}{\partial z} \right\rangle \quad \left. \vphantom{\frac{\partial \langle f_s \rangle}{\partial t}} \right\} \rightarrow \text{Slow evolution of Mean}$$

B. Fourier Transforms in Space & Time of ϕ_1 & f_{s1}

$$1. \text{Space: } \phi_1(z, t) = \int_{-\infty}^{\infty} dk \tilde{\phi}_1(k, t) e^{ikz}$$

$$f_{s1}(z, v_z, t) = \int_{-\infty}^{\infty} dk \tilde{f}_{s1}(k, v_z, t) e^{ikz}$$

$$2. \text{Time: } \tilde{\phi}_1(k, t) = \hat{\phi}_1(k) e^{-i\omega(k, \tau) t}$$

$$\tilde{f}_{s1}(k, v_z, t) = \hat{f}_{s1}(k, v_z) e^{-i\omega(k, \tau) t}$$

where frequency $\omega(k, \tau)$ may change on long timescale $\tau \equiv t/\epsilon^2$

C. Reality Condition

$$1. \hat{\phi}_1(k) = \hat{\phi}_1^*(-k)$$

$$2. \omega_r(k, \tau) = -\omega_r(-k, \tau)$$

$$\gamma(k, \tau) = \gamma(-k, \tau)$$

II. Derivation of Quasilinear Diffusion Equation (Continued)

A. Evolution of Mean Distribution

$$1. \frac{\partial \langle f_s \rangle}{\partial t} = \frac{q_s}{m_s} \frac{\partial}{\partial v_z} \left\langle f_{s1} \frac{\partial \phi_1}{\partial z} \right\rangle \quad \left. \vphantom{\frac{\partial \langle f_s \rangle}{\partial t}} \right\} \rightarrow \text{Let's calculate this using solutions for } f_{s1} \text{ \& } \phi_1.$$

$$\int_{\partial V_z} \left\langle f_{s1} \frac{\partial \phi_1}{\partial z} \right\rangle = \frac{\partial}{\partial z} \frac{1}{2L} \int_{-L}^L dz f_{s1} \frac{\partial \phi_1}{\partial z}$$

3. Substituting in spatial Fourier Transform \tilde{f}_{s1} and $\tilde{\phi}_1$,

$$= \frac{\partial}{\partial z} \frac{1}{2L} \int_{-L}^L dz \left[\int_{-\infty}^{\infty} dk \tilde{f}_{s1}(k, v_z, t) e^{ikz} \right] \left[\frac{\partial}{\partial z} \left[\int_{-\infty}^{\infty} dk' \tilde{\phi}_1(k', t) e^{ik'z} \right] \right]$$

4. Collecting all terms dependent on z: $= \int_{-\infty}^{\infty} dk' ik' \tilde{\phi}_1(k', t) e^{ik'z}$

$$= \frac{\partial}{\partial z} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \frac{ik'}{2L} \tilde{f}_{s1}(k, v_z, t) \tilde{\phi}_1(k', t) \left[\int_{-L}^L dz e^{i(k+k')z} \right]$$

5. NOTE: Identity: $\delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(x-x_0)} d\omega$

a. Thus, in the limit of large box size $L \rightarrow \infty$ (necessary for chosen boundary conditions), we have

$$\lim_{L \rightarrow \infty} \int_{-L}^L dz e^{i(k+k')z} = 2\pi \delta(k+k')$$

6. The δ -function can be used to evaluate the k' integral at $k' = -k$:

$$= \frac{\partial}{\partial z} \int_{-\infty}^{\infty} dk \frac{2\pi}{2L} (-ik) \tilde{f}_{s1}(k, v_z, t) \tilde{\phi}_1(-k, t) = -\frac{\pi}{L} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} dk ik \tilde{\phi}_1(-k, t) \tilde{f}_{s1}(k, v_z, t)$$

7. Now, let's substitute Time Fourier Transform $\hat{\phi}_1$ & \hat{f}_{s1}

$$g_1 = -\frac{\pi}{L} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} dk ik \left[\hat{\phi}_1(k) e^{-i\omega(k, \tau)t} \right] \left[\hat{f}_{s1}(k, v_z) e^{-i\omega(k, \tau)t} \right]$$

Lecture #12 (Continued)
 II. A.7. (Continued)

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$$b. = -\frac{\pi}{L} \frac{\partial}{\partial v_z} \int_{-\infty}^{\infty} dk ik \hat{\phi}_1(-k) \hat{f}_{s1}(k, v_z) e^{-i[\omega(-k, \tau) + \omega(k, \tau)]t}$$

8. NOTE! $\omega(-k, \tau) + \omega(k, \tau) = -\omega_p(k, \tau) + i\gamma(k, \tau) + \omega_p(k, \tau) + i\gamma(k, \tau) = 2i\gamma(k, \tau)$

9. Thus $\frac{\partial}{\partial v_z} \left\langle f_{s1} \frac{\partial \phi_1}{\partial v_z} \right\rangle = -\frac{\pi}{L} \frac{\partial}{\partial v_z} \int_{-\infty}^{\infty} dk ik \hat{\phi}_1(k) \hat{f}_{s1}(k, v_z) e^{+2i\gamma(k, \tau)t}$

10. Now, we'll substitute the solution for $\hat{f}_{s1} = -\frac{q_s}{m_s} \frac{k \hat{\phi}_1(k)}{(\omega - kv_z)} \frac{\partial \langle f_{s1} \rangle}{\partial v_z}$
 (see III C.3. of Lect #11) \rightarrow

$$= + \frac{\pi q_s}{L m_s} \frac{\partial}{\partial v_z} \int_{-\infty}^{\infty} dk ik^2 \frac{\hat{\phi}_1(-k) \hat{\phi}_1(k)}{\omega - kv_z} \frac{\partial \langle f_{s1} \rangle}{\partial v_z} e^{2i\gamma(k, \tau)t}$$

NOTE: Does not depend on k .

11. We can use $\hat{E}_1(k) = -ik \hat{\phi}_1(k)$ so eliminate $\hat{\phi}_1$:

$$= \frac{\pi}{L} \frac{q_s}{m_s} \frac{\partial}{\partial v_z} \left[\int_{-\infty}^{\infty} dk \frac{i \hat{E}_1(k) \hat{E}_1(-k)}{\omega - kv_z} e^{2i\gamma(k, \tau)t} \right] \frac{\partial \langle f_{s1} \rangle}{\partial v_z}$$

B. Writing in terms of Spectral Energy Density of Electric Field

1. The Electrostatic energy density is given by

$$W_E = \frac{\epsilon_0 |E_1(z, t)|^2}{2}$$

a. Averaging this energy density gives $\langle W_E \rangle = \frac{\epsilon_0}{4L} \int_{-L}^L dz |E_1(z, t)|^2$

II B. (Continued)

2. We can write $\langle W_E \rangle$ in terms of $\tilde{E}_1(k, t)$ by

$$\begin{aligned} \langle W_E \rangle &= \frac{\epsilon_0}{4L} \int_{-L}^L dz \left[\int_{-\infty}^{\infty} dk \tilde{E}_1(k, t) e^{ikz} \right] \left[\int_{-\infty}^{\infty} dk' \tilde{E}_1^*(k', t) e^{-ik'z} \right] \\ &= \frac{\epsilon_0}{4L} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \tilde{E}_1(k, t) \tilde{E}_1^*(k', t) \int_{-L}^L dz e^{i(k-k')z} \\ &= \frac{\pi \epsilon_0}{2L} \int_{-\infty}^{\infty} dk \tilde{E}_1(k, t) \tilde{E}_1^*(k, t) \end{aligned}$$

$\int_{-L}^L dz e^{i(k-k')z} = 2\pi \delta(k-k')$

3. Thus

$$\langle W_E \rangle = \int_{-\infty}^{\infty} dk \mathcal{E}(k, t)$$

where we define the Spectral Energy Density

$$\mathcal{E}(k, t) = \frac{\pi \epsilon_0}{2L} |\tilde{E}_1(k, t)|^2$$

4. We may write the time dependence of $\mathcal{E}(k, t)$ as

a. $\mathcal{E}(k, t) = \frac{\pi \epsilon_0}{2L} |\hat{E}_1(k)|^2 e^{2\gamma(k, t) +}$

b. This implies

$$\frac{d\mathcal{E}(k, t)}{dt} = 2\gamma(k, t) \mathcal{E}(k, t)$$

γ is treated as a constant

C. Putting it all together

1. We have $\frac{\partial \langle A \rangle}{\partial t} = \frac{\pi q_s^2}{k m_s^2} \frac{\partial}{\partial v_z} \left\{ \int_{-\infty}^{\infty} dk \left[\frac{i 2k}{\omega - kv_z} \mathcal{E}(k, t) \right] \frac{\partial \langle A \rangle}{\partial v_z} \right\}$

where we note $\hat{E}_1(k) \hat{E}_1(-k) = \hat{E}_1(k) \hat{E}_1^*(k) = |\hat{E}_1(k)|^2$

III C. (Continued)

2. finally we obtain

$$\frac{\partial \langle f_s \rangle}{\partial t} = \frac{\partial}{\partial v_z} \left[D_q(v_z, t) \frac{\partial \langle f_s \rangle}{\partial v_z} \right]$$

Quasilinear
Diffusion Equation

where the quasilinear Diffusion Coefficient is

$$D_q(v_z, t) = \frac{2}{\epsilon_0} \left(\frac{q_s}{m_s} \right)^2 \int_{-\infty}^{\infty} dk \frac{i \epsilon(k, t)}{\omega - kv_z}$$

D. Reality of Quasilinear Diffusion Coefficient

NOTE: $\frac{i \epsilon(k, t)}{\omega - kv_z} = \frac{i \epsilon(\omega_r - kv_z - i\gamma)}{(\omega_r + i\gamma - kv_z)(\omega_r - i\gamma - kv_z)}$

$$= \frac{i \epsilon(\omega_r - kv_z) + \epsilon \gamma}{(\omega_r - kv_z)^2 + \gamma^2}$$

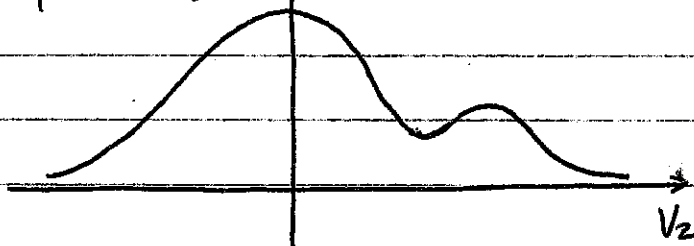
2. $\int_{-\infty}^{\infty} dk \frac{i \overbrace{\epsilon(k, t)}^{\text{Even}} \underbrace{(\omega_r - kv_z)}_{\text{odd}} + \overbrace{\epsilon(k, t)}^{\text{Even}} \underbrace{\gamma}_{\text{Even}}}{\underbrace{(\omega_r - kv_z)^2 + \gamma^2}_{\text{even}}}$

a. Since first term is overall odd, it cancels on integration over $\int_{-\infty}^{\infty} dk$.

b. Thus, we are left with a real quantity:

$$D_q(k, t) = \frac{2}{\epsilon_0} \left(\frac{q_s}{m_s} \right)^2 \int_{-\infty}^{\infty} dk \frac{\epsilon(k, t) \gamma(k, t)}{(\omega_r(k, t) - kv_z)^2 + \gamma^2(k, t)}$$

III. Application of Quasilinear Theory:

1. Initial System $f_0(v_z)$ f_0 

2. Three quantities must be evolved:
- $\langle f_s(v_z, \tau) \rangle$ Mean Distribution
 - $E(k, t, \tau)$ Spectral Energy Density
 - $\gamma(k, \tau)$ Growth Rate

3. Evolution Equations:

$$a. \frac{\partial \langle f_s \rangle}{\partial \tau} = \frac{\partial}{\partial v_z} \left[D_q(v_z, t, \tau) \frac{\partial \langle f_s \rangle}{\partial v_z} \right]$$

$$\text{where } D_q(v_z, t, \tau) = \frac{2}{\epsilon_0} \left(\frac{q_s}{m_s} \right)^2 \int_{-\infty}^{\infty} dk \frac{E(k, t, \tau) \gamma(k, \tau)}{(\omega_r(k, \tau) - kv_z)^2 + \gamma(k, \tau)^2}$$

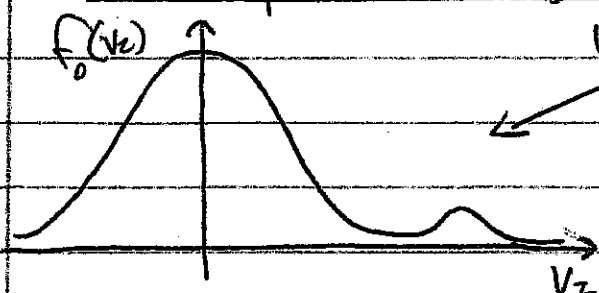
$$b. \frac{\partial E(k, t, \tau)}{\partial t} = 2 \gamma(k, \tau) E(k, t, \tau)$$

$$c. D(k, \omega) = 1 - \sum_s \frac{q_s n_s^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\frac{\partial \langle f_s \rangle}{\partial v_z}}{v_z - \frac{\omega}{k}} = 0$$

The solution of the dispersion relation yield $\gamma(k, \tau)$ for unstable modes.

Thus, $\langle f_s(v_z, \tau) \rangle$, $E(k, t, \tau)$, and $\gamma(k, \tau)$ must be advanced in time self-consistently.

II. (Continued)

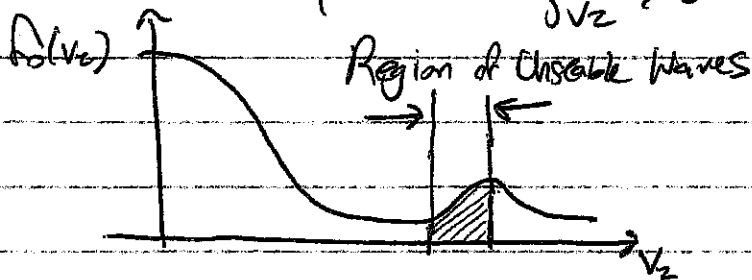
B. The Bump-on-Tail Instability

1. Initial Distribution

2. Assume $|\gamma| \ll |\omega_r|$ \Rightarrow Weak Growth Rates

3. Remember from Lec # 8 II.C.5., in the weak growth approximation,

$$a. \gamma = \pi \frac{k}{|k|} \frac{1}{\partial D_r / \partial \omega_r} \approx \frac{\omega_{ps}^2}{S} \frac{\partial f_{s0}}{\partial v_z} \Big|_{v_z = \frac{\omega_r}{k}}$$

b. Thus $\gamma > 0$ only where $\frac{\partial f_{s0}}{\partial v_z} > 0$ (for $v_z > 0$)

4. Since $\frac{\partial \mathcal{E}(k, t, \tau)}{\partial t} = 2\gamma(k, \tau) \mathcal{E}(k, t, \tau)$, the spectral energy density $\mathcal{E}(k)$ only grows in this range of unstable waves with $v_z = \frac{\omega}{k}$.

5. a. For a small, unstable growth rate $|\gamma| \ll |\omega_r|$, we

can estimate $\frac{\gamma}{(\omega_r - kv_z)^2 + \gamma^2} \approx \pi \delta(\omega_r - kv_z) = \frac{\pi}{v_z} \delta\left(\frac{\omega_r}{v_z} - k\right)$

b. We can thus evaluate $D_q(v_z, t, \tau)$

$$D_q(v_z, t, \tau) = \frac{2}{\epsilon_0} \left(\frac{q_s}{m_s}\right)^2 \int_{-\infty}^{\infty} dk \mathcal{E}(k, t, \tau) \frac{\pi}{v_z} \delta\left(\frac{\omega_r}{v_z} - k\right) = \frac{2\pi (q_s)^2}{\epsilon_0 (m_s)^2} \int_{-\infty}^{\infty} \frac{\mathcal{E}(k = \frac{\omega_r}{v_z})}{v_z} dk$$

Diffusion due to resonant waves

Lecture #12 (Continued)

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III. B. (Continued)

6. Therefore

$$a. \frac{\partial \langle P_s \rangle}{\partial t}(v_z, T) = \frac{\partial}{\partial v_z} \left\{ \frac{2\pi (q_s)^2}{\epsilon_0 (m_s)^2} \sum_{k=\frac{\omega_r}{v_z}} \frac{\partial P_s}{\partial v_z} \right\}$$

For each v_z , only unstable wave solutions of Dispersion relation with $v_z = \frac{\omega_r}{k}$ lead to diffusion.

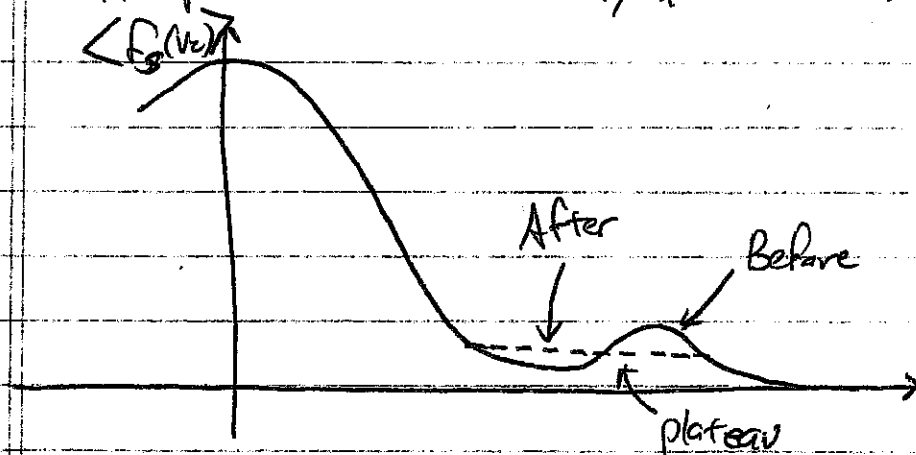
$$b. \text{The equation is of the form: } \frac{\partial f}{\partial t} = \frac{\partial}{\partial v_z} \left[K \frac{\partial f}{\partial v_z} \right] \sim K \frac{\partial^2 f}{\partial v_z^2}$$

This diffusion in velocity space!

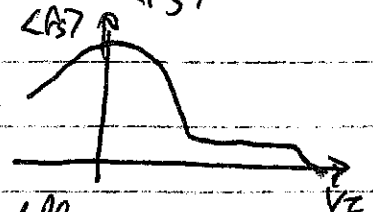
c. This diffusion serves to smooth out the distribution function in velocity space in regions where unstable waves exist, i.e. $\frac{\partial f}{\partial v_z} > 0$.

7. The Result

a. Since particles must be conserved, area under curve must not change.



b. Since diffusion occurs only in regions where $\frac{\partial \langle f_s \rangle}{\partial v_z} > 0$, the effect of quasilinear diffusion is to evolve $\langle f_s \rangle$ to a state where $\frac{\partial \langle f_s \rangle}{\partial v_z} \leq 0$ everywhere (for $v_z > 0$) \Rightarrow



c. At this point, you reach marginal stability, unstable waves are no longer generated, and quasilinear diffusion stops.