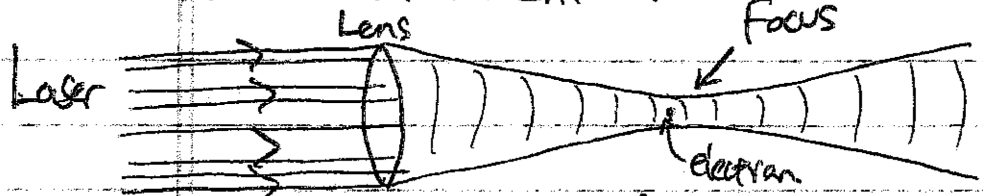
Lecture #17 The Ponderomotive Force

I. Particle Motion in High Frequency Electromagnetic Wave

A. Laser Plasma Interaction:



1. What is the motion of an electron in a high-frequency electromagnetic wave with variation in wave amplitude over space, ie., near the focal point of a laser?

2. In this case, the plasma is unmagnetized. Only $\underline{E}$ and $\underline{B}$ from wave are present.

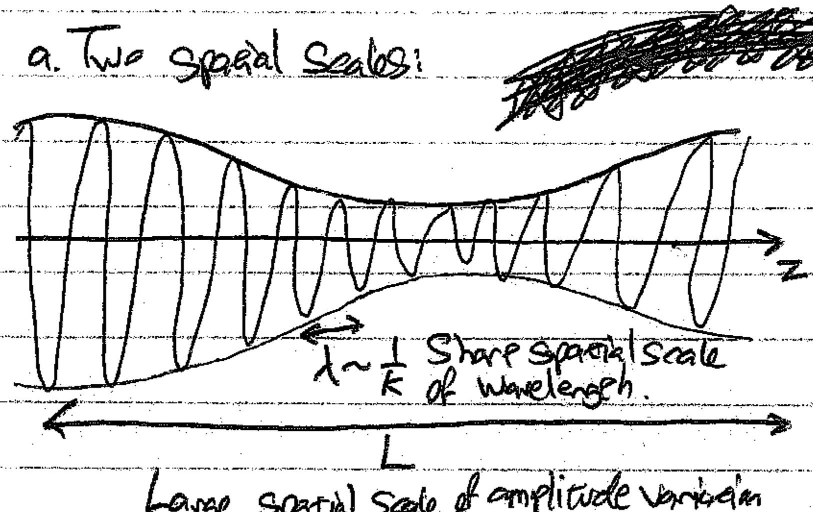
3. We'll use multiple timescale analysis to determine the lowest order nonlinear effect.

B. Multiple Timescale Analysis:

1. Consider an electromagnetic wave of high frequency ω whose amplitude may vary on a long timescale and large spatial scale.

$$\underline{E}(\underline{x},t) = \underline{E}_0(\underline{x},\tau)\cos(\omega t - \underline{k}\cdot\underline{x})$$

a. Two spatial scales:



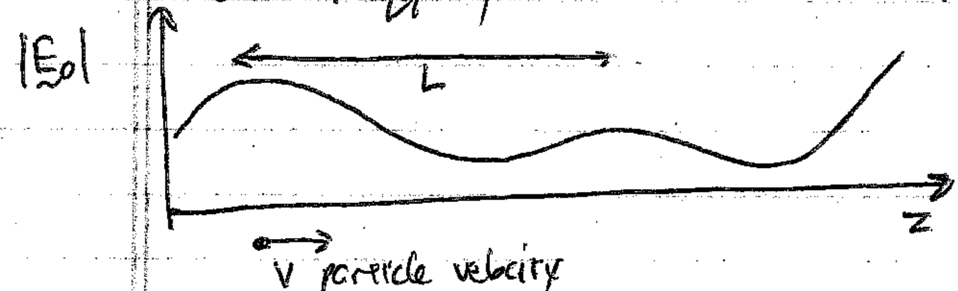
Lecture #17 (Continued) Howes ②

I.B. (Continued)

2. We'll show in HW that $\underset{\sim}{E}(\underset{\sim}{x}, T, t)$ yields

$$\underset{\sim}{B}(\underset{\sim}{x}, T, t) = -\frac{1}{\omega}\left\{ \nabla \times \underset{\sim}{E}_0(\underset{\sim}{x}, T) \sin(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x}) - \underset{\sim}{k} \times \underset{\sim}{E}_0^{(x,T)} \cos(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x}) \right\}$$

3. We want a ~~slow~~ slow variation of EM wave magnitude due to motion of the particle through space compared to the wave frequency ω!



a. To particle, amplitude varies in time due to motion $\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0$

$$\nabla \sim \frac{1}{L} \text{ large spatial scale} \qquad |\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0| \sim \frac{v}{L} E_0$$

b. Frequency of EM wave gives $|\omega \underset{\sim}{E}_0| \sim \omega E_0$

c. We want $\underbrace{|\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0|}_{\text{Slow timescale}} \ll \underbrace{|\omega\underset{\sim}{E}_0|}_{\text{Fast timescale}} \Rightarrow \frac{v}{L} \ll \omega$ or $\boxed{\frac{v}{L\omega} \ll 1}$

d. This will be our ordering parameter

$$\boxed{\epsilon \sim \frac{v}{L\omega} \ll 1}$$

This separates fast oscillation timescale due to EM wave from slow drift timescale due to amplitude variation

4. $\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})$ Compare these terms:

$$\frac{|\underset{\sim}{v}\times\underset{\sim}{B}|}{|\underset{\sim}{E}|} \sim \frac{|v||B|\cancel{\sin\theta}^{1}}{|E|} \sim \frac{v\left(\frac{E_0}{L\omega}\right)}{\cancel{E_0}} \sim \frac{v}{L\omega} \sim \epsilon \ll 1$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla\times\underset{\sim}{E} \Rightarrow \omega B \sim \frac{E_0}{L} \text{ or } B \sim \frac{E_0}{L\omega}$$

Lecture 17 (Continued)

I.B. (Continued)

5. Two Timescales: a. t Fast oscillation timescale

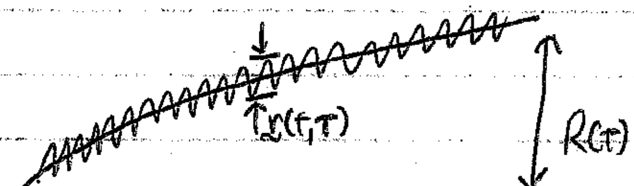
   b. $\tau = \epsilon t$ Slow timescale of amplitude variation

   c. Thus $\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}}$

   d. Over each fast oscillation period, $\underset{\sim}{E}$ varies little in amplitude. But small changes each oscillation can sum to produce a long timescale change.

6. Write particle position as slowly varying oscillation center $\underset{\sim}{R}(\tau)$ plus small, rapidly oscillating position $\underset{\sim}{r}(t,\tau)$

$$\boxed{\underset{\sim}{X} = \underset{\sim}{R}(\tau) + \epsilon \underset{\sim}{r}(t,\tau)}$$



7. Velocity:

$$\underset{\sim}{V} = \frac{dx}{dt} = \frac{d\underset{\sim}{R}(\tau)}{dt} + \epsilon \frac{d\underset{\sim}{r}(t,\tau)}{dt}$$

Define $\underset{\sim}{U} \equiv \frac{d\underset{\sim}{R}(\tau)}{dt} = \epsilon \frac{\partial \underset{\sim}{R}(\tau)}{\partial \tau}$

$\underset{\sim}{u} \equiv \frac{d\underset{\sim}{r}(t,\tau)}{dt},$

$$\underset{\sim}{V} = \epsilon \underset{\sim}{U}(\tau) + \epsilon \underset{\sim}{u}(t,\tau)$$

8. Acceleration:

$$\frac{d\underset{\sim}{v}}{dt} = \epsilon \frac{d\underset{\sim}{U}(\tau)}{dt} + \epsilon \frac{d\underset{\sim}{u}(t,\tau)}{dt} = \cancel{\epsilon \frac{\partial \underset{\sim}{U}(\tau)}{\partial t}}^{\,0} + \epsilon^2 \frac{\partial \underset{\sim}{U}(\tau)}{\partial \tau} + \epsilon \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial \tau}$$

9. Thus, we find:

$$\epsilon \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{U}(\tau)}{\partial \tau} + \epsilon^2 \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial \tau} = \epsilon \frac{q}{m}\left[\underset{\sim}{E}(\underset{\sim}{R}+\epsilon\underset{\sim}{r}, \tau, t) + \left(\epsilon \underset{\sim}{U}(\tau) + \epsilon \underset{\sim}{u}(t,\tau)\right) \times \underset{\sim}{B}(\underset{\sim}{R}+\epsilon\underset{\sim}{r}, \tau, t)\right]$$

a. NOTE that highest order nonzero term of LHS is $\mathcal{O}(\epsilon)$. Thus, highest term on RHS must be $\mathcal{O}(\epsilon)$ to balance. Hence, we multiply RHS by ϵ to give balance

Lecture #17 (Continued)

I.B. (Continued)

10. Taylor Expand Fields about oscillation center $\underset{\sim}{R}$:

a. $\underset{\sim}{E}(\underset{\sim}{x},\tau,t) = \underset{\sim}{E}(\underset{\sim}{R},\tau,t) + (\underset{\sim}{x}-\underset{\sim}{R})\cdot\nabla \underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \frac{[(\underset{\sim}{x}-\underset{\sim}{R})\cdot\nabla]^2}{2!}\underset{\sim}{E}(\underset{\sim}{R},\tau,t)+\dots$

b. NOTE: $\underset{\sim}{x}-\underset{\sim}{R} = \underset{\sim}{r}$, so

$$\underset{\sim}{E}(\underset{\sim}{x},\tau,t) = \underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \epsilon\, \underset{\sim}{r}\cdot\nabla\underset{\sim}{E}(\underset{\sim}{R},\tau,t)+\dots$$

and likewise with $\underset{\sim}{B}(\underset{\sim}{x},\tau,t)$

11. Expand all variables and substitute:

a. $\underset{\sim}{U}(\tau) = \underset{\sim}{U}_1(\tau) + \epsilon\, \underset{\sim}{U}_2(\tau)+\dots$

$\underset{\sim}{u}(\tau,t) = \underset{\sim}{u}_1(\tau,t) + \epsilon\, \underset{\sim}{u}_2(\tau,t)+\dots$

$\underset{\sim}{r}(\tau,t) = \underset{\sim}{r}_1(\tau,t) + \epsilon\, \underset{\sim}{r}_2(\tau,t)+\dots$

12. Thus, we get

$$\epsilon\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial t} + \epsilon^2\frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t} + \epsilon^2\frac{\partial \underset{\sim}{U}_1(\tau)}{\partial\tau} + \epsilon^3\frac{\partial \underset{\sim}{U}_2(\tau)}{\partial\tau} + \epsilon^2\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial\tau} + \epsilon^3\frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial\tau}$$

$$= \frac{q}{m}\Big[\epsilon\,\underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \epsilon^2\underset{\sim}{r}_1\cdot\nabla\underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \epsilon^2\underset{\sim}{U}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^3\underset{\sim}{U}_2\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)$$

$$+\epsilon^3\underset{\sim}{U}_1\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^4\underset{\sim}{U}_2\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^2\underset{\sim}{u}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)$$

$$+\epsilon^3\underset{\sim}{u}_2\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^3\underset{\sim}{u}_1\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^4\underset{\sim}{u}_2\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t)+\dots\Big]$$

13. Lowest Order: $O(\epsilon)$

$$\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial t} = \frac{q}{m}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{R})$$

$$\frac{\partial \underset{\sim}{r}_1}{\partial t} = \underset{\sim}{u}_1(t,\tau) = \frac{q}{m\omega}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R}) \longleftarrow \text{Oscillation velocity}$$

$$\underset{\sim}{r}_1(t,\tau) = \frac{-q}{m\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R}) \longleftarrow \text{Oscillation position}$$

Lecture #17 (Continued) Howes (5)

I. B. (Continued)

14. ~~Next~~ Order: $O(\epsilon^2)$

a. $$\underbrace{\frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t}}_{①} + \underbrace{\frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}}_{②} + \underbrace{\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial \tau}}_{③} = \frac{q}{m}\Big[\underbrace{\underset{\sim}{r}_1\cdot\nabla\underset{\sim}{E}(\underset{\sim}{R},\tau,t)}_{④} + \underbrace{\underset{\sim}{U}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)}_{⑤} + \underbrace{\underset{\sim}{u}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)}_{⑥}\Big]$$

b. We now average over oscillation period to annihilate terms $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} dt$.

i) Assume $\underset{\sim}{u}_2(t,\tau)$ is periodic over $T=\frac{2\pi}{\omega}$.

This should be checked a posteriori.

c. Term ①: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t}dt = 0 \text{ by assumed periodicity}$$

d. Term ②: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}dt = \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}\cancelto{1}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} dt} = \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}$$

e. Term ③: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial}{\partial \tau}\Big[\frac{q}{m\omega}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big]dt$$

$$= \frac{q}{m\omega}\frac{\partial \underset{\sim}{E}_0(\underset{\sim}{R},\tau)}{\partial \tau}\,\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt = 0$$

f. Term ④: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\frac{q}{m}\Big[\frac{-q}{m\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big]\cdot\nabla\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt$$

$$= \frac{-q^2}{m^2\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cdot\nabla\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\,\frac{\omega}{2\pi}\underbrace{\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt}_{=\frac{\pi}{\omega}} = \frac{-q^2}{2m^2\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cdot\nabla\underset{\sim}{E}_0(\underset{\sim}{R},\tau)$$

g. Term ⑤: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\frac{q}{m}\underset{\sim}{U}_1(\tau)\times\Big[\frac{-1}{\omega}\Big\{\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})-\underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big\}\Big]dt$$

$$= \frac{-q}{m\omega}\Big[\underset{\sim}{U}_1(\tau)\times\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cancelto{0}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt} - \underset{\sim}{U}(\tau)\times\underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cancelto{0}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})dt}\Big]$$

$$= 0$$

Lecture #7 (Continued) Howes ⑥

I. B.4 (Continued)

h. Term ⑥: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{q}{m}\left[\frac{q}{m\omega}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\right]\times\left\{\frac{-1}{\omega}\left[\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\right.\right.$

$\left.\left.-\underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\right]\right\}dt$

$$= \frac{-q^2}{m^2\omega^2}\left\{\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\times\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\underbrace{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\,dt}_{\to\frac{\pi}{\omega}}\right.$$

$$\left.-\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\times\underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\underbrace{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\,dt}_{\to 0}\right\}$$

$$= -\frac{q^2}{2m^2\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\times\nabla\times E_0(\underset{\sim}{R},\tau)$$

i. Putting solution together, we find

$$\frac{\partial \underset{\sim}{U}_1(\tau)}{\partial\tau} = -\frac{q^2}{2m^2\omega^2}\left[\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cdot\nabla E_0(\underset{\sim}{R},\tau)+\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\times\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\right]$$

j. NOTE! NRL p. 4 (12) gives

$$\nabla(\underset{\sim}{A}\cdot\underset{\sim}{B}) = \underset{\sim}{A}\times(\nabla\times\underset{\sim}{B})+\underset{\sim}{B}\times(\nabla\times\underset{\sim}{A})+(\underset{\sim}{A}\cdot\nabla)\underset{\sim}{B}+(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{A}$$

Taking $\underset{\sim}{A}=\underset{\sim}{B}=\underset{\sim}{E}$, we get $\nabla\left(\frac{|\underset{\sim}{E}|^2}{2}\right)=\underset{\sim}{E}\times(\nabla\times\underset{\sim}{E})+(\underset{\sim}{E}\cdot\nabla)\underset{\sim}{E}$

k. Thus, we find:

$$\boxed{\frac{\partial \underset{\sim}{U}_1(\tau)}{\partial\tau} = \frac{-q^2}{2m^2\omega^2}\nabla\left(\frac{|E_0(\underset{\sim}{R},\tau)|^2}{2}\right)}$$

Ponderomotive Force

Lecture #17 (Continued)

I. (Continued)

C. Properties of the Ponderomotive Force

1. $$\boxed{\underset{\sim}{F}_{pond} = m \frac{d\underset{\sim}{U}}{dt} = \frac{-q^2}{4m\omega^2} \nabla |\underset{\sim}{E}_0|^2}$$ Pushes away from regions of intense field.

a. We can write this as a potential force

$$\underset{\sim}{F}_{pond} = -\nabla \Phi_{pond}$$

where $$\boxed{\Phi_{pond} = \frac{q^2}{4m\omega^2}|\underset{\sim}{E}_0|^2}$$

b. Note that the average fast oscillation ~~velocity~~ energy is

$$\frac{1}{2} m \overline{|\underset{\sim}{u}|^2} = \frac{m}{2}\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{q^2}{m^2\omega^2}|\underset{\sim}{E}_0|^2 \underbrace{\sin^2(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{R})}_{=\frac{\pi}{\omega}} dt = \frac{m}{2}\frac{\omega}{2\pi}\frac{q^2}{m^2\omega^2}|\underset{\sim}{E}_0|^2\frac{\pi}{\omega}$$

$$= \frac{q^2}{4m\omega^2}|E_0|^2$$

c. Thus $$\Phi_{pond} = \frac{1}{2} m \overline{|\underset{\sim}{u}|^2}$$

Ponderomotive potential is the average fast oscillation kinetic energy.

d. For center of oscillation $$\boxed{\mathcal{E}_{osc} = \frac{1}{2} m U^2 + \Phi_{pond} = \frac{1}{2} m U^2 + \frac{1}{2} m \overline{|\underset{\sim}{u}|^2}}$$

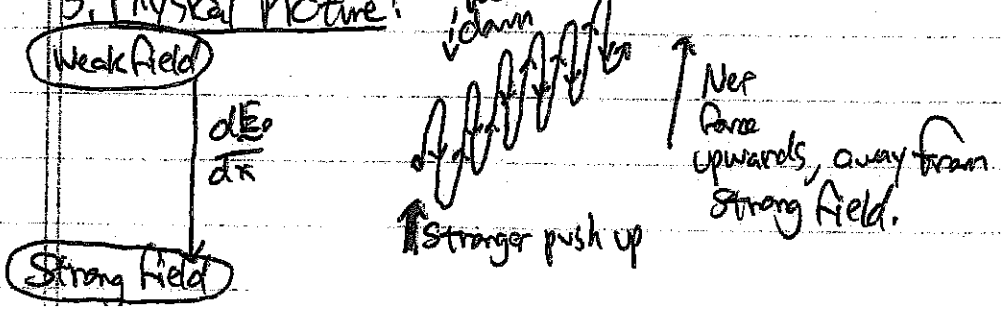
2. a. Force is independent of ~~sign of~~ charge

⇒ Repels both ions and electrons from high field regions.

b. Because $m_e \ll m_i$, electrons are pushed aside much more easily.

⇒ Resulting polarization electric field acts to pull ions out.

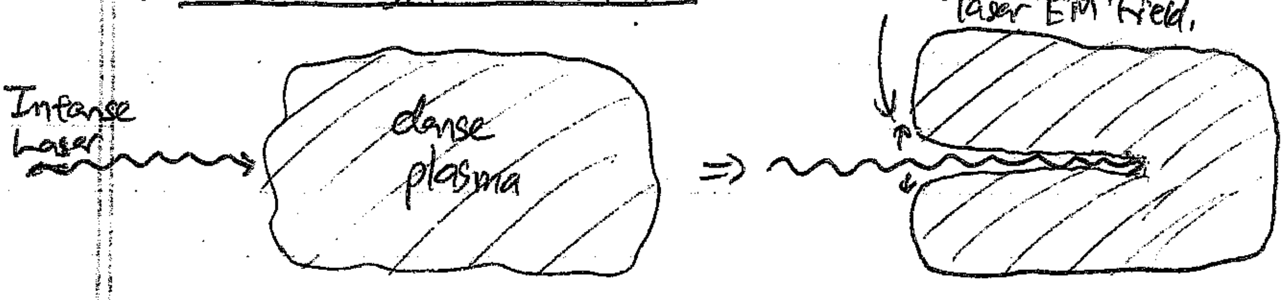
3. Physical Picture:



Lecture #17 (Continued) Hayes ⑧

I (Continued)

4. Laser Plasma Interaction:

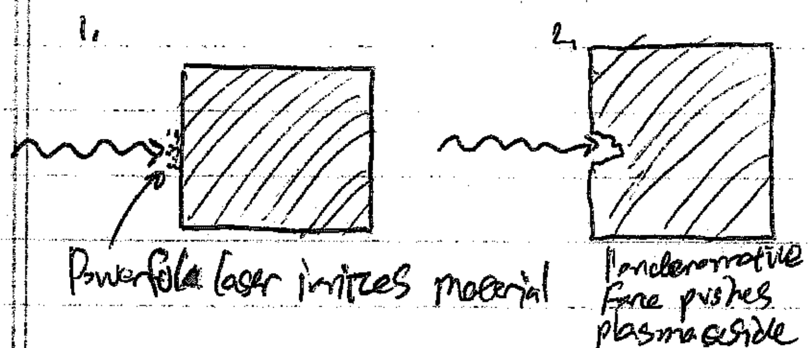


a. This can lead to self-focusing of laser light in a plasma.

1. Powerful laser pushes aside electrons (and ions) due to ponderomotive force.

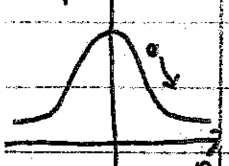
2. The resulting depression in plasma density acts as a convex lens, focusing the laser light into the evacuated channel.

b. Lasers can bore holes in materials by this mechanism.



3. Laser can propagate if light frequency $\omega > \omega_{pe}$ in plasma.

5. Example: A particle of charge q & mass m is initially at rest at the center of a Gaussian laser beam with $|\underset{\sim}{E_0}(\underset{\sim}{x})| = E_0 e^{-\frac{r^2}{r_0^2}}$. Find linear of oscillation velocity as a function of position.



$$\mathcal{E}_0 = \frac{1}{2} m u^2 + \Phi_{pond} \quad \text{where } \Phi_{pond} = \frac{q^2}{4m\omega^2}|\underset{\sim}{E_0}|^2 = \frac{q^2}{4m\omega^2} E_0^2 e^{-\frac{2r^2}{r_0^2}}$$

At $r=0$, $\mathcal{E}_0 = \frac{1}{2} m \cancel{u^2}^{\,0} + \frac{q^2}{4m\omega^2} E_0^2$.

Thus $$u = \sqrt{\frac{2\mathcal{E}_0}{m} - \frac{2\Phi_{pond}}{m}} = \sqrt{\frac{2\mathcal{E}_0}{m} - \frac{2\mathcal{E}_0}{m} e^{-\frac{2r^2}{r_0^2}}} = \sqrt{\frac{2\mathcal{E}_0}{m}}\left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}$$

$$\boxed{u(r) = \frac{q E_0}{m\omega\sqrt{2}}\left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}}$$