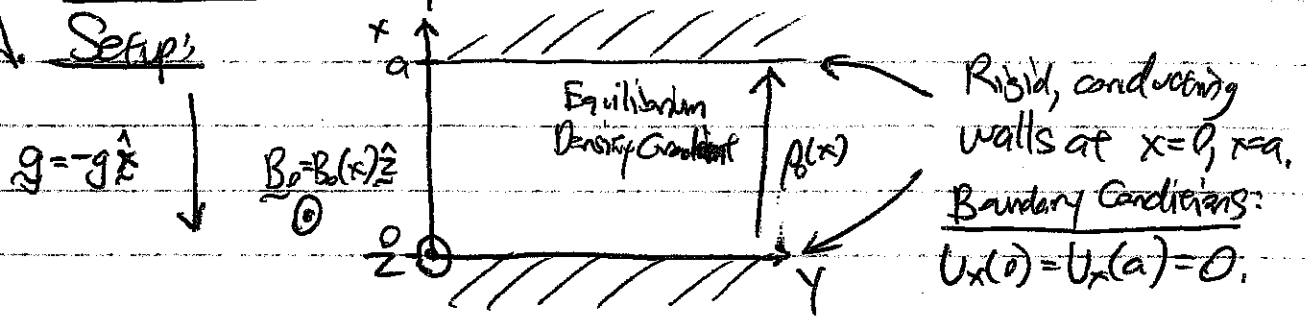


Lesson #3: MHD Stability Analysis of Rayleigh-Taylor Instability

I. Normal Mode Analysis

A. Setup:



1. Density has exponential form in direction of gravity (\hat{x})

a. $\rho_0(x) = \rho_{00} e^{-\frac{x}{H}}$ $H \equiv$ Scale height of density

b. For $H > 0$, density decreases with height (stable)

$H < 0$, density increases with height (unstable)

2. For simplicity, we assume $\frac{\partial}{\partial z} = 0$ (No variation along mean field).

NOTE: Such variations would bend the magnetic field lines, leading to magnetic tension (which stabilizes instability)

b. We also assume incompressible motion, $\nabla \cdot \underline{U}_1 = 0$

3. We want to analyze this problem for ~~form~~ stability.

a. Normal Mode Analysis

b. Energy Principle

B. Using Linear Force Operator

1. We could use $-\rho_0 \omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$

to solve for the characteristic frequencies. But instead, we'll begin from equation of motion.

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1.5. Using Equation of Motion:

1. Momentum Eq:

$$\rho \frac{\partial \underline{U}}{\partial t} + \rho \underline{U} \cdot \nabla \underline{U} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \frac{(\underline{B} \cdot \nabla) \underline{B}}{\mu_0} + \rho \underline{g}$$

gravity, where $\underline{g} = -g \hat{z}$

2. Linearize about Equilibrium:

$$\rho = \rho_0(x) + \epsilon \rho_1(x)$$

$$\underline{U} = \underline{U}_0(x) + \epsilon \underline{U}_1(x)$$

$$\underline{B} = B_0(x) \hat{z} + \epsilon \underline{B}_1(x)$$

$$p = p_0(x) + \epsilon p_1(x)$$

vertical direction (scalar)

vector

3. Lowest Order: $\mathcal{O}(1)$: $0 = -\nabla \left(p_0 + \frac{B_0^2}{2\mu_0} \right) + \rho_0 \underline{g}$

a. Equilibrium satisfies: $\frac{\partial}{\partial x} \left(p_0 + \frac{B_0^2}{2\mu_0} \right) = -\rho_0 g$ Static Equilibrium

b. Notation: $p_0' = \frac{\partial p_0}{\partial x}$, $B_0' = \frac{\partial B_0}{\partial x}$, $\rho_0' = \frac{\partial \rho_0}{\partial x}$

$$\Rightarrow \boxed{p_0' + \frac{B_0 B_0'}{\mu_0} = -\rho_0 g}$$

4. Next Order: $\mathcal{O}(\epsilon)$:

$$\rho_0 \frac{\partial \underline{U}_1}{\partial t} = -\nabla \left(p_1 + \frac{B_0 \cdot \underline{B}_1}{\mu_0} \right) + \frac{(\underline{B}_0 \cdot \nabla) \underline{B}_1}{\mu_0} - \frac{\underline{B}_1 \cdot \nabla \underline{B}_0}{\mu_0} + \rho_1 \underline{g}$$

①
②
③
④
⑤

a. Term ③ $(\underline{B}_0 \cdot \nabla) \underline{B}_1 = B_0 \frac{\partial}{\partial x} \underline{B}_1 = 0$

b. Term ④ $(\underline{B}_1 \cdot \nabla) \underline{B}_0 = B_{1x} \frac{\partial B_0 \hat{z}}{\partial x} = B_0' B_{1x} \hat{z}$

c. We can eliminate Term ② by taking the curl of this equation: $(\nabla \times \nabla \phi = 0)$.
↑
 scalar function.

$$d. \nabla \times \left(\rho_0 \frac{\partial \underline{U}_1}{\partial t} \right) = \nabla \times (B_0' B_x \hat{z}) + \nabla \times (-\rho_1 g \hat{x})$$

e. Note: Since $\frac{\partial}{\partial z} = 0$, $\nabla = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y}$

f. Let's find the \hat{z} -component!

$$1. \hat{z} \cdot \left[\nabla \times \left(\rho_0 \frac{\partial \underline{U}_1}{\partial t} \right) \right] = \frac{\partial}{\partial x} \left[\rho_0 \frac{\partial U_{1y}}{\partial t} \right] - \frac{\partial}{\partial y} \left[\rho_0 \frac{\partial U_{1x}}{\partial t} \right]$$

$$= \rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t}$$

$$2. \hat{z} \cdot \left[\nabla \times (B_0' B_x \hat{z}) \right] = 0$$

$$3. \hat{z} \cdot \left[\nabla \times (-\rho_1 g \hat{x}) \right] = -\frac{\partial}{\partial y} [-\rho_1 g] = g \frac{\partial \rho_1}{\partial y}$$

g. Thus, we find $\boxed{\rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t} = g \frac{\partial \rho_1}{\partial y}}$

5. Fourier Transform in y and t : $U_1(x) = \underline{U}_1(x) e^{i(k_y y - \omega t)}$

a. Thus $\frac{\partial}{\partial t} = -i\omega$ $\frac{\partial}{\partial y} = ik_y$

b. This yields

$$\boxed{\rho_0' U_y + \rho_0 \frac{\partial U_y}{\partial x} - \rho_0 ik_y U_x = -\frac{k_y g \rho_1}{\omega}}$$

6. We assume incompressible motion $\nabla \cdot \underline{U}_1 = 0$

$$\boxed{\frac{\partial U_x}{\partial x} + ik_y U_y = 0}$$

$$\Rightarrow U_y = -\frac{i}{k_y} \frac{\partial U_x}{\partial x}$$

7. Continuity Equation $\frac{\partial \rho}{\partial t} + \underline{U} \cdot \nabla \rho + \rho \nabla \cdot \underline{U} = 0$

a. $\mathcal{O}(\epsilon)$: $\frac{\partial \rho_1}{\partial t} + \underline{U}_1 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \underline{U}_1 = 0$

b. $\boxed{-i\omega \rho_1 + U_x \rho_0' = 0} \Rightarrow \rho_1 = -\frac{i}{\omega} U_x \rho_0'$

8. Eliminate U_y & p_1 in favor of U_x :

a. This yields:

$$\frac{\partial^2 U_x}{\partial x^2} + \frac{\rho_0'}{\rho_0} \frac{\partial U_x}{\partial x} - k_y^2 \left[1 + \frac{g}{\omega^2} \left(\frac{\rho_0'}{\rho_0} \right) \right] U_x = 0$$

9. Note: Since $\rho_0(x) = \rho_0 e^{-\frac{x}{H}}$, we have $\frac{\rho_0'}{\rho_0} = -\frac{1}{H}$

$$\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H} \frac{\partial U_x}{\partial x} - k_y^2 \left(1 - \frac{g}{\omega^2 H} \right) U_x = 0$$

10. We can solve this with the help of an integrating factor.

a. Let $U_x(x) = f(x) e^{\frac{x}{2H}}$

b. This gives $\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H} \frac{\partial U_x}{\partial x} = \left(\frac{\partial^2 f}{\partial x^2} - \frac{f}{4H^2} \right) e^{\frac{x}{2H}}$

c. Thus, we find:

$$\frac{\partial^2 f}{\partial x^2} + \alpha^2 f = 0 \quad \text{where } \alpha^2 = k_y^2 \left(\frac{g}{H\omega^2} - 1 \right) - \frac{1}{4H^2}$$

11. The function $f(x)$ must satisfy the boundary conditions

$$U_x(0) = U_x(a) = 0 \quad \Rightarrow \quad f(0) = f(a) = 0$$

a. General Solution: $f(x) = f_0 \sin \alpha x + f_1 \cos \alpha x$

b. $f(0) = f_0(0) + f_1(1) = 0 \Rightarrow f_1 = 0$.

c. $f(a) = f_0 \sin(\alpha a) = 0 \Rightarrow \alpha = \frac{n\pi}{a}$ for $n=1, 2, 3, \dots$

d. We therefore have eigenfunctions f_n with mode number n .

12. Solve for frequency: $\frac{n^2 \pi^2}{a^2} = k_y^2 \left(\frac{g}{H\omega_n^2} - 1 \right) = \frac{1}{4H^2}$

a. $\frac{n^2 \pi^2}{a^2} + k_y^2 + \frac{1}{4H^2} = \frac{g}{H} \frac{k_y^2}{\omega_n^2}$

$$b. \omega_n^2 = \left(\frac{g}{H} \right) \frac{4H^2 k_y^2 a^2}{a^2 + 4H^2 (n^2 \pi^2 + k_y^2 a^2)}$$

positive definite

c. For $H > 0$, $\omega_n^2 > 0 \Rightarrow$ Oscillating function \Rightarrow STABLE
 For $H < 0$, $\omega_n^2 < 0 \Rightarrow$ UNSTABLE.

d.1. Max growth rate $\omega_n^2 = \frac{g}{H}$ as $k_y \rightarrow \infty$

2. Growth rate $\rightarrow 0$ as $k_y \rightarrow 0$

3. Lower vertical mode numbers n have faster growth.

II. Energy Principle:

A. Gravitational Force Term:

1. In Linear Force Operator $\underline{F}(\underline{\xi})$, we must add gravity term:

$$+ \rho_1 g$$

a. From leave #1, II. A.4, b, 2, we have $\rho_1 = -\underline{\xi} \cdot \nabla \rho_0 = \rho_0 \nabla \cdot \underline{\xi} = 0$

b. For incompressible motion, $\nabla \cdot \underline{\xi} = 0$, so $\rho_1 = -\underline{\xi} \cdot \nabla \rho_0 = \underline{\xi}_x \rho_0'$

c. Thus $+ \rho_1 g = (-\underline{\xi}_x \rho_0') (-g \hat{x}) = \rho_0' g \underline{\xi}_x \hat{x}$

2. This gives:

$$\underline{F}(\underline{\xi}) = \nabla \left[\underline{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{\xi} \right] + \frac{(\nabla \times \underline{B}_0) \times [\nabla \times (\underline{\xi} \times \underline{B}_0)] + (\nabla \times [\nabla \times (\underline{\xi} \times \underline{B}_0)]) \times \underline{B}_0}{\mu_0} + \rho_0' g \underline{\xi}_x \hat{x}$$

3. For the energy principle, we must add this term:

$$-\frac{1}{2} |\underline{\xi}_x|^2 g \rho_0'$$

[Remember, $\delta W = -\frac{1}{2} \int d^3x \underline{\xi} \cdot \underline{F}(\underline{\xi})$]

II. (Continued)

B. Using Energy Principle

1. With the added gravitational potential term, we have

$$\delta W = \frac{1}{2} \int d^3x \left\{ \frac{|\nabla \times (\underline{\xi} \times \underline{B}_0)|^2}{\mu_0} + \rho_0 |\nabla \cdot \underline{\xi}|^2 - \underline{\xi}^* \cdot \underline{j}_0 \times [\nabla \times (\underline{\xi} \times \underline{B}_0)] - \underline{\xi}^* \cdot \nabla (\underline{\xi} \cdot \nabla p_0) - |\underline{\xi}_x|^2 g \rho_0' \right\}$$

2. a. $\nabla \times (\underline{\xi} \times \underline{B}_0) = \underline{\xi} (\nabla \cdot \underline{B}_0) - \underline{B}_0 (\nabla \cdot \underline{\xi}) + (\underline{B}_0 \cdot \nabla) \underline{\xi} - (\underline{\xi} \cdot \nabla) \underline{B}_0$
NRL (10) p. 4

$$= -B_0' \underline{\xi}_x \hat{z}$$

b. $\underline{j}_0 = \frac{\nabla \times \underline{B}_0}{\mu_0} = \frac{1}{\mu_0} \left(\frac{\partial}{\partial x} \hat{y} - \frac{\partial}{\partial y} \hat{x} \right) \wedge (B_0 \hat{z}) = \frac{1}{\mu_0} B_0' \hat{y}$

3. TERM ①: $= \frac{(B_0')^2}{\mu_0} |\underline{\xi}_x|^2$

4. TERM ③: $= -\underline{\xi}_x^* \cdot \left(-\frac{B_0'}{\mu_0} \hat{y} \right) \times \left(-B_0' \underline{\xi}_x \hat{z} \right) = \underline{\xi}_x^* \cdot \left(\frac{B_0'^2}{\mu_0} \underline{\xi}_x \hat{x} \right) = -\frac{(B_0')^2}{\mu_0} |\underline{\xi}_x|^2$

a. Thus, Term ① + Term ③ = 0 ✓

5. Term ④: $-\underline{\xi}^* \cdot \nabla \left[\underbrace{(\underline{\xi} \cdot \nabla) p_0}_{\text{Scalar} = f} \right]$

a. NOTE: NRL

(7) p. 4 $\nabla \cdot (f \underline{A}) = f \nabla \cdot \underline{A} + \underline{A} \cdot \nabla f$

$$\nabla \cdot (\underline{\xi}^* f) = f \nabla \cdot \underline{\xi}^* + \underline{\xi}^* \cdot \nabla f$$

b. Thus $\int d^3x \left\{ \cancel{\underline{\xi}^* \cdot \nabla [(\underline{\xi} \cdot \nabla) p_0]} - \underline{\xi}^* \cdot \nabla [(\underline{\xi} \cdot \nabla) p_0] \right\} = \int d^3x \nabla \cdot \left[\underline{\xi}^* (\underline{\xi} \cdot \nabla) p_0 \right]$

By Divergence Thm $\int d^3x \nabla \cdot \left[\underline{\xi}^* (\underline{\xi} \cdot \nabla) p_0 \right] = \int dS \cdot \left[\underline{\xi}^* \underline{\xi}_x p_0' \right] = 0$

NRL (28) p. 5

By B.C.'s $\underline{\xi}_x$ at boundary $\neq 0$, p_0' is zero!
 Periodic in y & z sums to zero.

6. Term ⑤: Only term left:

$$\delta W = -\frac{1}{2} \int d^3x |\underline{\xi}_z|^2 g \rho_0'$$

Lecture #3 (Continued)

Pages ⑦

II B. (Continued)

7. Thus, for $\rho_0' > 0$ (density increasing with height),

$\delta W < 0 \Rightarrow$ UNSTABLE!