Lecture #5: Kinetic Theory for Electrostatic Waves in Unmagnetized Plasma

I. Review of Kinetic Theory

A. The Boltzmann Equation (Plasma Kinetic Equation)

1. $$\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \underbrace{\frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}}}_{\substack{\text{Lorentz Force gives} \\ \text{acceleration}}} = \left(\frac{\partial f_s}{\partial t}\right)_{coll}$$

where $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ is the distribution function in 6-D phase space.

2. a. Recall from 029:194 Lecture #11, ratio of collisional to collective effects in a plasma,

$$\frac{\text{collisional effects}}{\text{collective effects}} \sim \frac{\nu_c}{\omega_{pe}} \sim \frac{1}{N_D} \ll 1$$

$\underset{\uparrow}{\text{Number of particles in}}$
the Debye sphere.

b. Thus, for many plasmas, we can neglect the effect of collisions,

$$\left(\frac{\partial f}{\partial t}\right)_{coil} = 0$$

Thus, Boltzmann Equation $\Rightarrow$ Vlasov Equation

B. Vlasov-Maxwell System of Equations

1. The starting point for our studies of kinetic theory is this system.

a. Vlasov Equation $\quad \frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0 \quad$ for $S = i, e$

b. Maxwell's Equations:

$\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\varepsilon_0} \quad$ Poisson $\qquad\qquad \rho_q = \sum_s \int d^3v\, q_s f_s$

$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \quad$ Faraday $\qquad\qquad \underset{\sim}{j} = \sum_s \int d^3v\, q_s \underset{\sim}{v} f_s$

$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \varepsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t} \quad$ Ampere-Maxwell

$\nabla \cdot \underset{\sim}{B} = 0$

c. This is a closed, integro-differential system of equations for 8 unknowns $f_i(\underset{\sim}{x}, \underset{\sim}{v}, t)$, $f_e(\underset{\sim}{x}, \underset{\sim}{v}, t)$, $\underset{\sim}{E}(\underset{\sim}{x}, t)$, $\underset{\sim}{B}(\underset{\sim}{x}, t)$

I.   (Continued)

C. The Distribution Function

1. Maxwellian Distribution

  a. For many linear problems, we take the lowest order (equilibrium) distribution function to be Maxwellian.

  b. Maxwellian distributions characterize <u>local Thermodynamic Equilibrium</u> (a maximum entropy state — no free energy)

  c.

$$f_{S_m}(\underline{x},\underline{v},t) = \frac{n_s(\underline{x},t)}{\pi^{\frac{3}{2}} V_{T_s}^3(\underline{x},t)} e^{-\frac{m_s|\underline{v}-\underline{U}_s(\underline{x},t)|^2}{2 T_s(\underline{x},t)}}$$

where <u>DEF</u>: Thermal Velocity $\quad V_{T_s}^2 \equiv \frac{2 T_s(\underline{x},t)}{m_s}$

> <u>NOTE</u>: As in Lecture #4, for the rest of the semester, we will absorb Boltzmann's constant $k = 1.38 \times 10^{-27} \frac{J}{K}$ into the temperature, $kT_s \Rightarrow T_s$, giving temperature T_s in units of energy (J).

  d. For $\overset{steady}{\wedge}$ uniform conditions (homogeneous in space, $\frac{\partial f_{sm}}{\partial \underline{x}} = 0$), and no flow velocity $\underline{U} = 0$, this simplifies to

$$\boxed{f_{sm}(v) = \frac{n_{s_0}}{\pi^{\frac{3}{2}} V_{T_s}^3} e^{-\frac{v^2}{V_{T_s}^2}}}$$

<u>NOTE</u>: This is a function of only $v = |\underline{v}|$.

2. Moments of the distribution function:

  a. Density : $\quad n_{s_0} = \int d^3\underline{v} \; f_{sm}(v)$

  b. Energy : $\quad \frac{3}{2} n_{s_0} T_{s_0} = \int d^3\underline{v} \; \frac{1}{2} m_s v^2 \, f_{sm}(v)$

I.C. (Continued)

3. Plot

V/V_{es}

Normalized by species thermal velocity

f_{sm}, $f_{i, ions}$, $f_{e, electrons}$

Equal area under curves for $n_{io} = n_{eo}$.
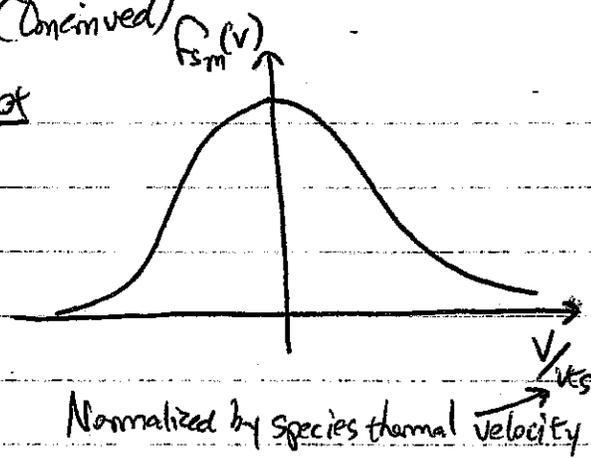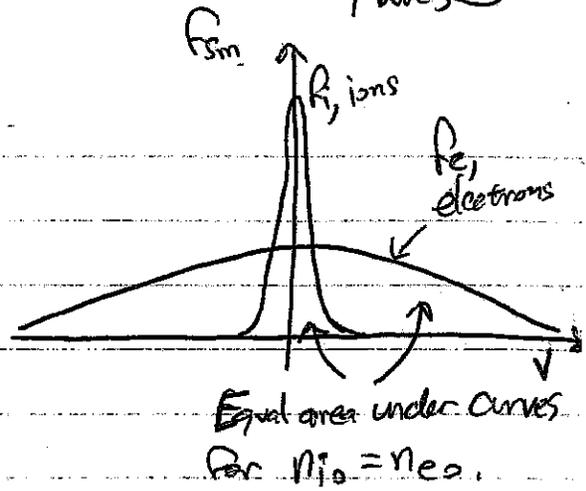
4. Reduced Distribution Function:

a. DEF: $F_s(v_z) \equiv \dfrac{1}{n_{so}} \displaystyle\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \, f_s(v)$

b. Integrate over two velocity space dimensions v_x & v_y.

c. For Maxwellian,

$$F_{s_m}(v_z) = \frac{1}{n_{so}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_x dv_y \left[\frac{n_{so}}{\pi^{\frac{3}{2}} v_{es}^{\frac{3}{2}}} e^{-\frac{(v_x^2 + v_y^2 + v_z^2)}{v_{es}^2}} \right]$$

$$= \frac{e^{-\frac{v_z^2}{v_{es}^2}}}{\pi^{\frac{1}{2}} v_{es}} \underbrace{\int_{-\infty}^{\infty} \frac{dv_x}{v_{es}} \frac{e^{-\frac{v_x^2}{v_{es}^2}}}{\pi^{\frac{1}{2}}}}_{=1} \underbrace{\int_{-\infty}^{\infty} \frac{dv_y}{v_{es}} \frac{e^{-\frac{v_y^2}{v_{es}^2}}}{\pi^{\frac{1}{2}}}}_{=1} = \boxed{\frac{e^{-\frac{v_z^2}{v_{es}^2}}}{\pi^{\frac{1}{2}} v_{es}} = F_{s_m}(v_z)}$$

d. NOTE: $\int_{-\infty}^{\infty} F_s(v_z) dv_z = 1$

II. Electrostatic Waves in an Unmagnetized Plasma

A. Setup

1. Electrostatic Approximation, $B_1 = 0 \Rightarrow \nabla \times E = 0 \Rightarrow E = -\nabla \phi$

2. No mean magnetic field, $B_0 = 0$, And $E_0 = 0$.

3. Vlasov-Maxwell System Simplifies to

a. $\dfrac{\partial f_s}{\partial t} + v \cdot \nabla f_s - \dfrac{q_s}{m_s} \nabla \phi \cdot \dfrac{\partial f_s}{\partial v} = 0$ (For ions & electrons)

b. $-\nabla^2 \phi = \dfrac{1}{\epsilon_0} \displaystyle\sum_s \int d^3v \, q_s f_s$

Closed System of Equations for $f_e(x, v, t)$, $f_i(x, v, t)$, $\phi(x, t)$

II. (Continued)

B. Linearization:

1. NOTE: Now $\underline{v}$ is a coordinate and not a variable. Therefore, we don't expand $\underline{v}$.

2. Take

a. $f_s = f_{s_0}(\underline{v}) + \epsilon f_{s_1}(\underline{x}, \underline{v}, t)$

b. $\phi(\underline{x}, t) = \phi_0^{\nearrow 0} + \epsilon \phi_1(\underline{x}, t)$

3. NOTE: a. In steady state, $\dfrac{\partial f_{s_0}}{\partial t} = 0$

b. For a homogeneous plasma, $\dfrac{\partial f_{s_0}}{\partial \underline{x}} = 0$.

c. Since $\underline{E}_0 = 0 \Rightarrow \phi_0 = 0$.

4. a. $\epsilon \dfrac{\partial f_{s_1}}{\partial t} + \epsilon \underline{v} \cdot \nabla f_{s_1} - \epsilon \dfrac{q_s}{m_s} \nabla \phi_1 \cdot \dfrac{\partial f_{s_0}}{\partial \underline{v}} - \epsilon^2 \dfrac{q_s}{m_s} \nabla \phi_1 \cdot \dfrac{\partial f_{s_1}}{\partial \underline{v}} = 0$

b. $-\epsilon \nabla^2 \phi_1 = \dfrac{1}{\epsilon_0} \sum_s \int d^3v \, q_s f_{s_0} + \dfrac{1}{\epsilon_0} \epsilon \sum_s \int d^3v \, q_s f_{s_1}$

5. NOTE: $\int d^3v \, q_s f_{s_0} = n_{s_0} q_s$, so the first term of RHS of Poisson's equations becomes $\dfrac{1}{\epsilon_0} \sum_s n_{s_0} q_s = 0$ for charge neutral equilibrium.

6. $O(\epsilon)$:

$$\boxed{\begin{array}{c} \dfrac{\partial f_{s_1}}{\partial t} + \underline{v} \cdot \nabla f_{s_1} - \dfrac{q_s}{m_s} \nabla \phi_1 \cdot \dfrac{\partial f_{s_0}}{\partial \underline{v}} = 0 \\[2mm] -\nabla^2 \phi_1 = \dfrac{1}{\epsilon_0} \sum_s \int d^3v \, q_s f_{s_1} \end{array}}$$

C. Fourier Transform in Space and Time

1. As usual, we'll solve this by Fourier transform $\sim e^{i(\underline{k} \cdot \underline{x} - \omega t)}$

$$\nabla \Rightarrow i\underline{k} \qquad \dfrac{\partial}{\partial t} \Rightarrow -i\omega$$

2. $-i\omega f_{s_1} + i \underline{v} \cdot \underline{k} f_{s_1} - i \dfrac{q_s \phi_1}{m_s} \underline{k} \cdot \dfrac{\partial f_{s_0}}{\partial \underline{v}} = 0$

$$k^2 \phi_1 = \dfrac{1}{\epsilon_0} \sum_s \int d^3v \, q_s f_{s_1}$$

Lecture #5

I. C. (Continued)

3. Solving for f_{s1}:

$$\boxed{f_{s1} = \dfrac{\dfrac{-q_s \phi_1}{m_s} \underset{\sim}{k} \cdot \dfrac{\partial f_{s0}}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}}}$$

4. Substituting f_{s1} into Poisson's Equation:

a. $k^2 \phi_1 = \dfrac{1}{\epsilon_0} \sum\limits_s \int d^3\underset{\sim}{v} \left(\dfrac{-q_s^2 \phi_1}{m_s} \dfrac{\underset{\sim}{k} \cdot \dfrac{\partial f_{s0}}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} \right)$

5. Dividing by k^2 and collecting terms:

$$\left[1 + \sum\limits_s \underbrace{\left(\dfrac{n_{s0} q_s^2}{\epsilon_0 m_s} \right)}_{\omega_{ps}^2} \dfrac{1}{k^2 n_{s0}} \int d^3\underset{\sim}{v} \dfrac{\underset{\sim}{k} \cdot \dfrac{\partial f_{s0}}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} \right] \phi_1 = 0$$

$$\underbrace{\hspace{8cm}}_{\text{Dispersion relation } D(\omega, \underset{\sim}{k})}$$

D. Simplifying

1. Take $\underset{\sim}{k} = k\hat{\underset{\sim}{z}}$ without loss of generality.

a. $\underset{\sim}{k} \cdot \dfrac{\partial f_{s0}}{\partial \underset{\sim}{v}} = k \dfrac{\partial f_{s0}}{\partial v_z}$

b. $\underset{\sim}{k} \cdot \underset{\sim}{v} = k v_z$

2. a. $D(\omega, \underset{\sim}{k}) = \left\{ 1 + \sum\limits_s \dfrac{\omega_{ps}^2}{k^2} \int\limits_{-\infty}^{\infty} dv_z \dfrac{k \dfrac{\partial}{\partial v_z}}{\omega - k v_z} \underbrace{\left[\dfrac{1}{n_{s0}} \int\limits_{-\infty}^{\infty} dv_x \int\limits_{-\infty}^{\infty} dv_y \, f_{s0} \right]}_{= F_{s0}(v_z)} \right\} = 0$

Reduced Distribution Function

b. $\boxed{D(\omega, k) = 1 - \sum\limits_s \dfrac{\omega_{ps}^2}{k^2} \int\limits_{-\infty}^{\infty} dv_z \dfrac{\partial F_{s0}/\partial v_z}{v_z - \dfrac{\omega}{k}} = 0}$

Dispersion Relation

E. Failure of Fourier Transform Approach

1. The integral above in $D(\omega, k)$ does not converge.

a. At $v_z = \dfrac{\omega}{k}$, denominator is zero.

b. Unless $\dfrac{\partial F_{s0}}{\partial v_z}$ (and F_{s0}) $= 0$ at $v_z = \dfrac{\omega}{k}$, integral does not converge.

2. The failure of $V_z = \frac{\omega}{k}$ occurs when there are particles with velocities that march the phase velocity of the wave.

$\Rightarrow$ These particles are <u>resonant</u> with the wave.

F. <u>The Bohm-Gross Dispersion Relation</u>

1. Let's consider a plasma of electrons with stationary ions forming a neutralizing background. $n_{io} = n_{eo}$, $f_{i_1} = 0$.

2. Cold Plasma Limit:

a. Assume electron thermal velocity much less than phase velocity.

$$V_{Te}^2 \simeq \langle V_z^2 \rangle \ll \frac{\omega^2}{k^2}$$

where $\langle V_z^2 \rangle = \int_{-\infty}^{\infty} dv_z \, V_z^2 \, F_{e_0}(V_z)$

b. Strictly, we can only use this approach when $F_{e_0}'(V_z)\Big|_{V_z = \frac{\omega}{k}} = 0$ (no resonant particles)

3a. Integrate $D(\omega, k)$ by parts

$$\int_{-\infty}^{\infty} dv_z \, \frac{\partial F_{s_0}/\partial v_z}{V_z - \frac{\omega}{k}} = \left[\frac{F_{s_0}}{V_z - \frac{\omega}{k}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dv_z \frac{F_{s_0}}{(V_z - \frac{\omega}{k})^2}$$

$u = \frac{1}{V_z - \frac{\omega}{k}}$ $dv = \frac{\partial F_{s_0}}{\partial v_z} dv_z$ $\lim_{V_z \to \pm\infty} F_{s_0}(V_z) = 0$

$du = \frac{-dv_z}{(V_z - \frac{\omega}{k})^2}$ $v = F_{s_0}$

b. Thus $D(\omega, \underline{k}) = 1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_{e_0}}{(V_z - \frac{\omega}{k})^2} = 0$

4. Expand denominator for $V_z \ll \frac{\omega}{k}$

$$\frac{1}{(V_z - \frac{\omega}{k})^2} = \frac{k^2}{\omega^2 (1 - \frac{k v_z}{\omega})^2} \simeq \frac{k^2}{\omega^2} \left[1 + 2\left(\frac{k v_z}{\omega}\right) + 3\left(\frac{k v_z}{\omega}\right)^2 + \cdots \right]$$

5. $D(\omega, k) = 1 - \frac{\omega_{pe}^2}{\omega^2} \int_{-\infty}^{\infty} dv_z \, F_{eo} \left[\underbrace{1}_{①} + \underbrace{2\left(\frac{kv_z}{\omega}\right)}_{②} + \underbrace{3\left(\frac{kv_z}{\omega}\right)^2}_{③} \right]$

a. $① = \int_{-\infty}^{\infty} dv_z \, F_{eo} = 1$

b. $② = 2 \int_{-\infty}^{\infty} dv_z \frac{k}{\omega} v_z \, F_{eo} = 0$ (odd in v_z)

c. $③ = 3 \frac{k^2}{\omega^2} \int_{-\infty}^{\infty} dv_z \, v_z^2 \, F_{eo}(v_z) \equiv \frac{3k^2}{\omega^2} \langle v_z^2 \rangle$ ✱

NOTE: The form of $\langle v_z^2 \rangle$ depends on equilibrium distribution function.

6. a. $D(\omega, k) = 1 - \frac{\omega_{pe}^2}{\omega^2} \left(✱ 1 + \frac{3 k^2 \langle v_z^2 \rangle}{\omega^2} \right) = 0$

b. $\boxed{\omega^2 = \omega_{pe}^2 \left(1 + 3 \frac{k^2 \langle v_z^2 \rangle}{\omega^2} \right)}$

c. NOTE: We have assumed $\frac{k^2 \langle v_z^2 \rangle}{\omega^2} \ll 1$, so second term is a small correction. This can be solved easily by the <u>method of successive approximations</u>.

7. <u>Method of Successive Approximations:</u>

a. Solve for ω^2 by dropping small term: $\omega^2 = \omega_{pe}^2 \left(1 + 3 \frac{k^2 \langle v_z^2 \rangle}{\omega^2} \right)^{0}$

   $\Rightarrow \omega_0^2 = \omega_{pe}^2$

b. Insert first solution ω_0^2 into small term to get second solution ω_1^2:

   $\omega_1^2 = \omega_{pe}^2 \left(1 + \frac{3 k^2 \langle v_z^2 \rangle}{\omega_0^2} \right) = \omega_{pe}^2 + 3 k^2 \langle v_z^2 \rangle$

   $\underset{= \omega_{pe}^2}{\longrightarrow}$

c. Thus, we find $\boxed{\omega^2 = \omega_{pe}^2 + 3 k^2 \langle v_z^2 \rangle}$

II. F. (Continued)

8. Alternative Explanation of Method of Successive Approximations

a. $\omega^2 = \omega_{pe}^2 \left(1 + \frac{3k^2\langle v_z^2\rangle}{\omega^2}\right)$ small term

b. Let $x = \omega^2$, $a = \omega_{pe}^2$, $b = 3k^2\langle v_z^2\rangle$ $\Rightarrow$ $x = a\left(1 + \epsilon\frac{b}{x}\right)$

c. $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

d. $x_0 + \epsilon x_1 = a\left[1 + \epsilon \frac{b}{(x_0 + \epsilon x_1)}\right] \simeq a\left[1 + \frac{\epsilon b}{x_0\left(1 + \frac{\epsilon x_1}{x_0}\right)}\right] \simeq a\left(1 + \frac{\epsilon b}{x_0} - \epsilon^2\frac{bx_1}{x_0^2}\right)$

e. $\mathcal{O}(1)$: $x_0 = a$ $\Rightarrow$ $\omega_0^2 = \omega_{pe}^2$

f. $\mathcal{O}(\epsilon)$: $x_1 = \frac{ab}{x_0}$ $\Rightarrow$ $\omega_1^2 = \frac{\omega_{pe}^2 \, 3k^2\langle v_z^2\rangle}{\omega_{pe}^2} = 3k^2\langle v_z^2\rangle$

g. $x = x_0 + \epsilon x_1$ $\Rightarrow$ $\boxed{\omega^2 = \omega_{pe}^2 + 3k^2\langle v_z^2\rangle}$

9. Maxwellian Equilibrium Distribution:

a. $\langle v_z^2\rangle = \int_{-\infty}^{\infty} dv_z \, v_z^2 \, F_{e_0}(v_z) = v_0^2 \int_{-\infty}^{\infty} \frac{dv_z}{v_{te}} \frac{v_z^2}{v_{te}^2} \frac{e^{-\frac{v_z^2}{v_{te}^2}}}{\pi^{\frac{1}{2}}} = v_{te}^2 \int_{-\infty}^{\infty} dy \, y^2 \frac{e^{-y^2}}{\pi^{\frac{1}{2}}}$

$y = \frac{v_z}{v_{te}}$

$= v_{te}^2 \frac{\sqrt{\pi}}{2\sqrt{\pi}} = \left(\frac{8 T_e}{m_e}\right)\frac{1}{8} = \frac{T_e}{m_e}$

b. Thus $\boxed{\omega^2 = \omega_{pe}^2 + 3k^2\frac{T_e}{m_e}}$ Langmuir Waves

c. From 029:194 Lec#24, $\omega^2 = \omega_{pe}^2 + \gamma k^2 C_e^2$ for Langmuir Waves

  1. DEF: Electron Sound Speed: $C_e^2 \equiv \frac{T_e}{m_e}$

  2. In Lecture #24, we took $\gamma_e = 1$ for isothermal conditions, which requires $v_{te} \gg \frac{\omega}{k}$. Here we are in the opposite limit, so we put γ_e back in.

10. Bohm-Gross Dispersion Relation

a. $\omega^2 = \omega_{pe}^2 + \gamma_e k^2 C_e^2$ Warm, Two-Fluid Theory $\left(\begin{array}{c}\text{Adiabatic}\\\text{(Fluid Closure assumed)}\end{array}\right)$

b. $\omega^2 = \omega_{pe}^2 + 3k^2 C_e^2$ Kinetic Theory

c. Results agree for $\gamma_e = 3$ (one degree of freedom; $\gamma = \frac{f+2}{f}$)

※※※ Kinetic Theory gives result <u>without</u> assuming an Equation of State!