

# Lecture #9 Review of Landau Damping; Cold Beam Instabilities Hawes ①

## I. Review: Landau Damping of Langmuir Waves

### A. Electrostatic Approximation: Vlasov-Maxwell System

$$1. \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

$$2. -\nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s \int d^3v q_s f_s$$

### B. Laplace Transform Approach by Landau

1. Linearization, Fourier Transform in Space, Laplace Transform in Time

$$\tilde{\phi}_1(\mathbf{k}, p) = \frac{N(\mathbf{k}, p)}{D(\mathbf{k}, p)} \quad \text{where } N(\mathbf{k}, p) = -i \sum_s \frac{q_s k_b}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{f_s(v_z)}{v_z - i\eta/k}$$

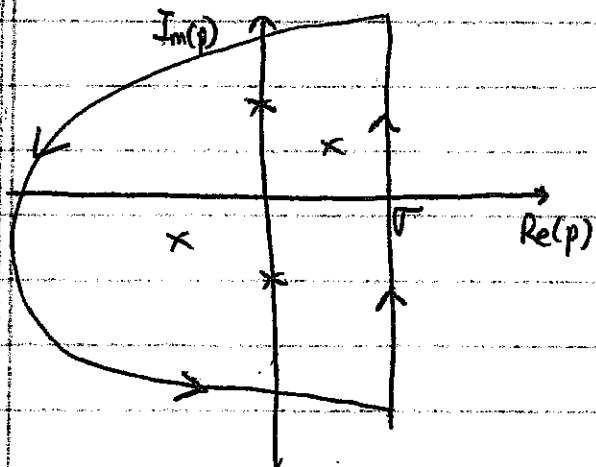
and the Dispersion Relation is  $D(\mathbf{k}, p) = 0$  where

$$D(\mathbf{k}, p) = 1 - \sum_s \frac{q_s k_b^2}{\epsilon_0 k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial f_s / \partial v_z}{v_z - i\eta/k}$$

$D(\mathbf{k}, p) = 0$  gives normal modes of system.

2. To solve, the Inverse Laplace Transform is

$$a. \phi_1(\mathbf{k}, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp \tilde{\phi}_1(\mathbf{k}, p) e^{pt}$$



b. We evaluate this complex contour integral by closing the contour at  $\text{Re}(p) \rightarrow -\infty$  and using the residue theorem.

c. To do so, we must analytically continue  $\tilde{\phi}_1(\mathbf{k}, p)$  from  $\text{Re}(p) > 0$  to  $\text{Re}(p) < 0$ .

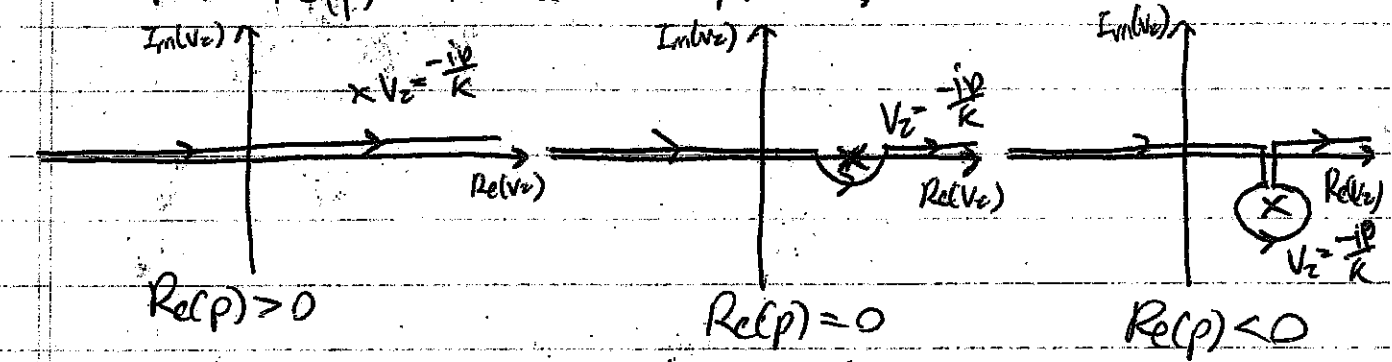
Lecture #9 (Continued)

Have 2

I.A. (Continued)

3. How do we analytically continue  $D(k, p) = 1 - \frac{c v_p^2}{\omega^2} \int_{-\infty}^{\infty} \frac{dv_z}{v_z - \frac{i p}{k}}$

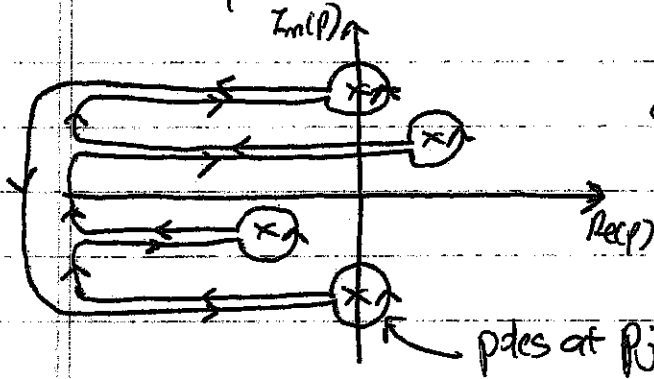
From  $\text{Re}(p) > 0$  to  $\text{Re}(p) < 0$ ?



⇒ Contour always passes below pole!

This definition yields an analytic form of  $\tilde{\phi}(k, p)$  over entire complex  $p$ -plane (except for poles, of course).

4. We may then evaluate the Inverse Laplace Transform by Residue Theorem



$$a. \phi_i(k, t) = \sum_j \text{Res} \left[ \tilde{\phi}_i(k, p) e^{pt} \right]$$

C. Landau Damping of Waves

1. For a Cauchy Distribution of electrons with stationary ions,

$$a. F_e(k, z) = \frac{C}{\pi} \left( \frac{1}{\omega + i v z} \right)$$

b. Dispersion Relation:  $D(k, p) = 1 + \frac{c v_p^2}{(p + i k |C|)^2} = 0$

c. Wave Solutions with  $\omega = \pm c v_p$  and  $\gamma = -|k| C$

↖ Landau damping due to  $\text{Re}(p) < 0$  (or  $\text{Im}(\omega) < 0$ )

I. C. (Continued)

2. Maxwellian Distribution:

a. The Plasma Dispersion Function

$$Z(\xi_s) = \int_C \frac{dz}{\pi^{1/2}} \frac{e^{-z^2}}{z - \xi_s}$$

Where  $\xi_s = \frac{ip}{k v_{Ts}}$  or  $\xi_s = \frac{\omega}{k v_{Ts}} + i \frac{\gamma}{k v_{Ts}}$

b. Using this function, the Langmuir Wave Dispersion Relation can be written,

$$D(k, p) = 1 + \frac{1}{k^2 \lambda_{De}^2} [1 + \xi_e Z(\xi_e)] = 0$$

c. In the high phase velocity limit,  $|\xi_e| \gg 1$  or  $\frac{\omega}{k} \gg v_{Te}$ , we find

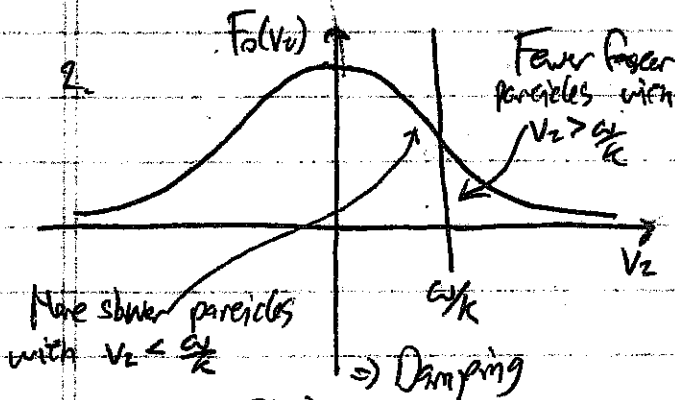
$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{Te}^2 \quad \text{and} \quad \gamma = -\frac{\sqrt{\pi}}{8} \frac{\omega_{pe}^3}{|k|^3 \lambda_{De}^3} e^{\left(\frac{-1}{2k^2 \lambda_{De}^2} - \frac{3}{2}\right)}$$

↑ Landau Damping

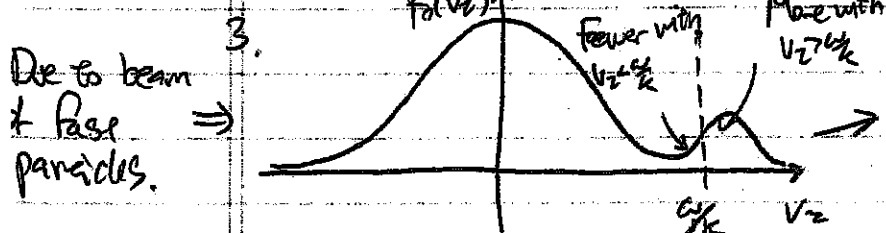
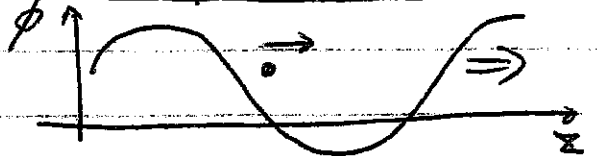
D. Physical Interpretation of Landau Damping

1. In the Weak Growth Rate Approximation,  $|\gamma| \ll |\omega|$ ,

$$\gamma = \pi \frac{k}{|k|} \frac{1}{\partial D_r / \partial \omega} \sum_s \frac{\omega_{ps}^2}{k^2} \left. \frac{\partial F_{s0}}{\partial v_z} \right|_{v_z = \frac{\omega}{k}}$$



↑  $v_z = \frac{\omega}{k}$   
Damping is due to Resonant particles in distribution func.  
→ Wave-particle Interaction

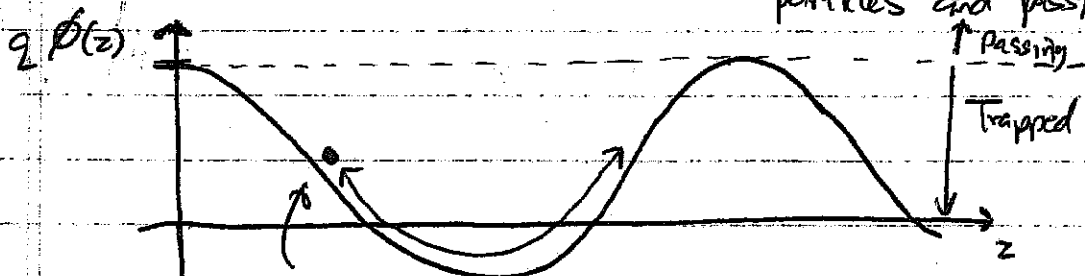
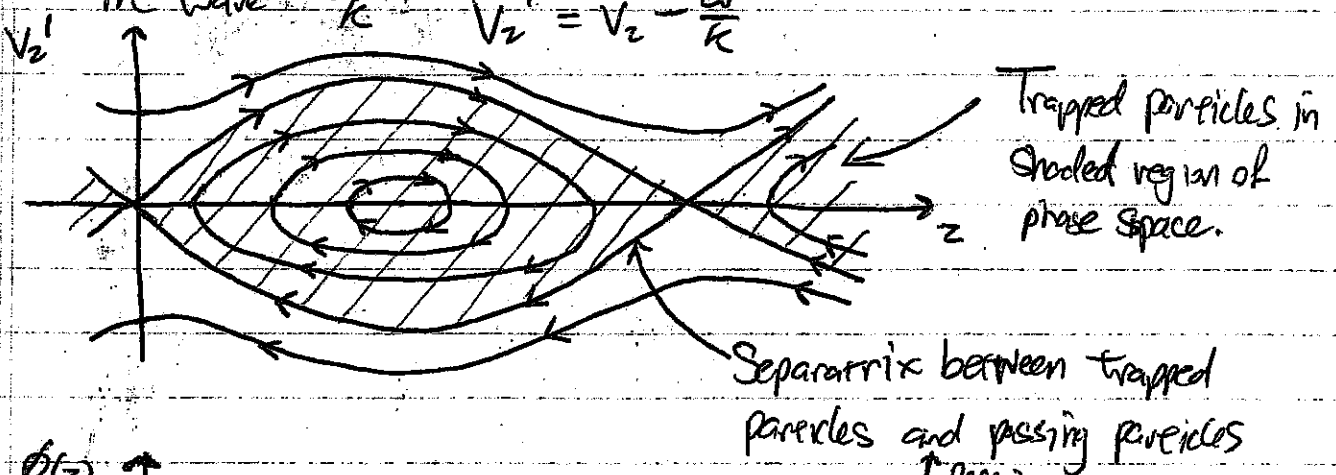


Bump-on-tail Instability:  
Leads to unstable growth of  $E$   
Fluctuations from free energy in  $F_0(v_z)$ .

II. Phase Mixing Interpretation

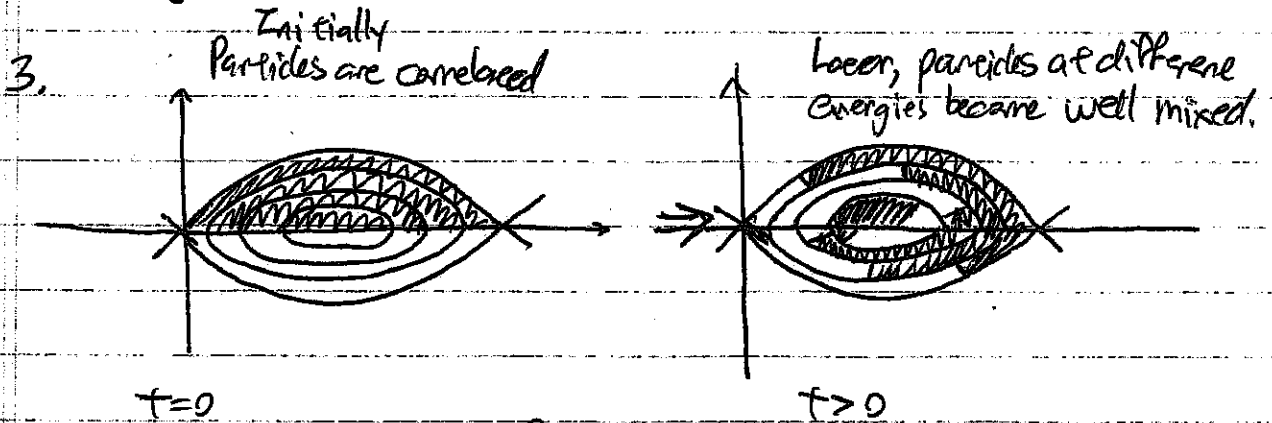
A. Phase Space Plot ( $v_z, z$ )

1. Transform to frame of reference moving at phase velocity of the wave  $\frac{\omega}{k}$ :  $v_z' = v_z - \frac{\omega}{k}$



Particles trapped in potential well (moving at phase speed  $\frac{\omega}{k}$ )

2. Bounce Frequency of trapped particles decreases with increasing energy, reaching zero at separatrix.



a. Since potential  $-D^2\phi = \frac{4\pi}{3} \rho^0 v_z$  depends on integration over  $f$ , this phase mixing leads to ~~an~~ averaging out of the integral  $\Rightarrow$  damping of the wave.

### III. Cold Beam Instabilities

#### A. Cold Beam Distribution:

1. A beam with number density  $n_j$  and velocity  $\underline{v} = v_j \hat{z}$  is given by  $F_0(\underline{v}) = n_j \delta(v_x) \delta(v_y) \delta(v_z - v_j)$

- Since all particles have the same velocity, the beam has zero temperature  $\Rightarrow$  no thermal spread of velocities.
- For a species  $s$ , we can have multiple beams,

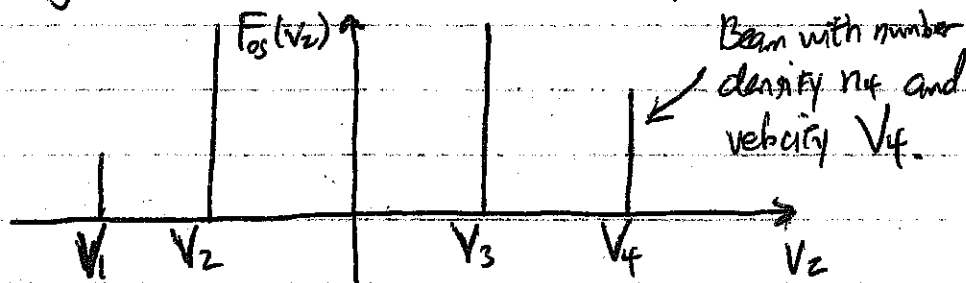
$$F_{0s}(\underline{v}) = \sum_j n_{js} \delta(v_x) \delta(v_y) \delta(v_z - v_{js})$$

4. The one-dimensional distribution function is written

$$F_{0s}(v_z) = \frac{1}{n_{0s}} \int_{-\infty}^{\infty} F_{0s}(\underline{v}) dv_x dv_y = \sum_j \frac{n_{js}}{n_{0s}} \delta(v_z - v_{js})$$

where  $n_0 = \sum_j n_j$  is the total number density of all beams.

5. Distribution:



#### B. Dispersion Relation for Electrostatic Waves

1. The Laplace-Fourier Solution for electrostatic waves gives a Dispersion Relation

$$D(k, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_{0s}}{(v_z - \frac{ip}{k})^2} = 0$$

2. Using  $\sum_j \omega_{ps}^2 \frac{n_{js}}{n_{0s}} = \sum_j \frac{n_{0s} q_s^2}{\epsilon_0 m_s} \frac{n_{js}}{n_{0s}} = \sum_j \omega_{pj}^2$  where  $\omega_{pj}^2 = \frac{n_j q_j^2}{\epsilon_0 m_j}$ ,

we obtain:

$$D(k, p) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \frac{1}{(v_{js} - \frac{ip}{k})^2} = 0$$

## III B (Continued)

3. NOTE: Here we take a complex  $\omega = i\gamma$

(we may write  $\omega = \omega_r + i\gamma$  to denote real & imaginary parts).

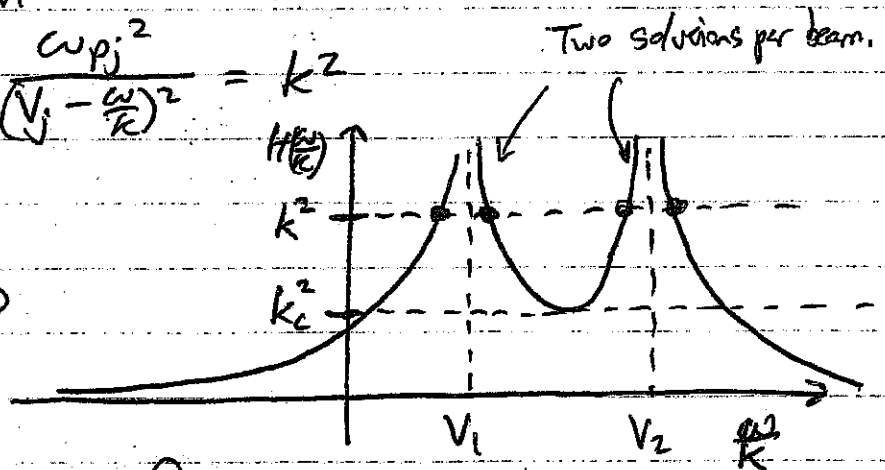
$$\text{So } D(k, \omega) = 1 - \sum_j \frac{\omega p_j^2}{k^2 (v_j - \frac{\omega}{k})^2} = 0$$

4. Solutions occur when

$$H\left(\frac{\omega}{k}\right) = \sum_j \frac{\omega p_j^2}{(v_j - \frac{\omega}{k})^2} = k^2$$

a. We may plot this as

For two beams  $\Rightarrow$



b. For  $k > k_c$ , there exist four real solutions  $\Rightarrow \gamma = 0$ .

c. But, for  $k < k_c$ , one obtains two real & two imaginary solutions.

Since  $H(\frac{\omega}{k})$  is real, the roots come in complex conjugate pairs

$$\omega_+ = \omega_r + i\gamma$$

$$\omega_- = \omega_r - i\gamma$$

} here we assume  $\gamma > 0$ .

d. Thus, the root  $\omega_+ = \omega_r + i\gamma$  gives a time dependence

$$\sim e^{-i\omega_r t + \gamma t}$$

Graph of this  $\omega_+$  solution.  $\Rightarrow$  UNSTABLE

5. a. Single beam leads to  $\omega = \pm \omega_p - kV \Rightarrow$  Always stable

b. When two (or more) beams are present, for sufficiently long wavelengths  $k < k_c$ , the plasma is always unstable.

Two Stream Instability

## III. (Continued)

C. Two Stream Instability: Equal and opposite beams.

1. Take  $n_1 = n_2 = n_0$       $V_1 = -V_2 = V$ .

2. Thus

$$D(k, \omega) = 1 - \frac{c_p^2}{(\omega - kV)^2} - \frac{c_p^2}{(\omega + kV)^2} = 0$$

$$c_p^2 = \frac{n_0 q_s^2}{\epsilon_0 m_s}$$

3. a. This equation is quadratic in  $\omega^2$  and can be expressed as

$$\omega^4 - \omega^2 2(c_p^2 + k^2 V^2) - k^2 V^2 (2c_p^2 - k^2 V^2) = 0$$

b. Solution: 
$$\omega^2 = \underbrace{c_p^2 + k^2 V^2}_{\text{Positive definite}} \pm \sqrt{\underbrace{c_p^4 + 4c_p^2 k^2 V^2}_{\text{Positive definite}}}$$

4. For solution with negative sign, when

$$\omega_p^2 + k^2 V^2 < (c_p^4 + 4c_p^2 k^2 V^2)^{\frac{1}{2}},$$

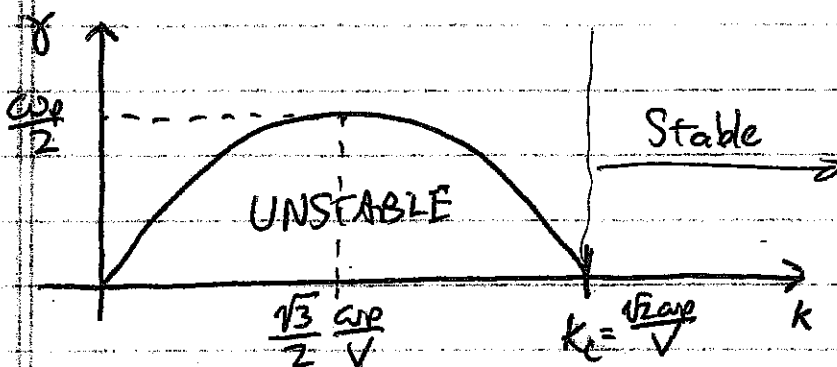
then  $\omega^2 < 0$  and the solution  $\omega = \pm i\gamma$ , leading to unstable growth.

b. Condition for this is

$$k < k_c = \sqrt{2} \frac{c_p}{V}$$

c. The growth rate is

$$\gamma = \left[ (c_p^4 + 4k^2 V^2 c_p^2)^{\frac{1}{2}} - (c_p^2 + k^2 V^2) \right]^{\frac{1}{2}}$$



## III. D. Weak Beam Approximation

1. Take a low density electron beam in a plasma at rest,  $n_b \ll n_0$ .2. For  $\omega_b^2 = \frac{n_b q^2}{\epsilon_0 m}$ , we get  $D(k, \omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\epsilon \omega_b^2}{(kV - \omega)^2} = 0$ 3. Ordering:  $\epsilon = \frac{\omega_b^2}{\omega_p^2} \ll 1$ : Let  $\omega = \omega_0 + \epsilon \omega_1$ .

a. 
$$\left[1 - \frac{\omega_p^2}{(\omega_0 + \epsilon \omega_1)^2}\right] (kV - (\omega_0 + \epsilon \omega_1))^2 = \epsilon \omega_b^2$$

b. Expanding:  $\frac{\omega_p^2}{\omega_0^2 (1 + \frac{\epsilon \omega_1}{\omega_0})^2} \approx \frac{\omega_p^2}{\omega_0^2} \left(1 - \frac{2\epsilon \omega_1}{\omega_0}\right)$

4. O(1):  $\left(1 - \frac{\omega_p^2}{\omega_0^2}\right) (kV - \omega_0)^2 = 0 \Rightarrow \omega_0 = \pm \omega_p \quad k = \frac{\pm \omega_p}{V}$

5. To solve at O( $\epsilon$ ), substitute solution  $\omega_0$  and  $k$ ,

a. 
$$\left[1 - \frac{\omega_p^2}{\omega_0^2} \left(1 - \frac{2\epsilon \omega_1}{\omega_0}\right)\right] [kV - \omega_0 - \epsilon \omega_1]^2 = \epsilon \omega_b^2$$

$$\Rightarrow \boxed{\omega_1^3 = \frac{\omega_b^2 \omega_p}{2}}$$

b. Three roots are:  $\omega_1 = \left(\frac{1}{2}\right)^{1/3} \omega_b^{2/3} \omega_p^{1/3}$  and  $\omega_1 = \left(\frac{1}{2}\right)^{1/3} \omega_b^{2/3} \omega_p^{1/3} \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$

6. Root with positive sign gives unstable growth with

a.  $\omega_r = \omega_p \left[1 - \frac{1}{2} \left(\frac{1}{2}\right)^{1/3} \left(\frac{n_b}{n_0}\right)^{1/3}\right]$  and  $\gamma = \frac{\sqrt{3}}{2} \left(\frac{1}{2}\right)^{1/3} \left(\frac{n_b}{n_0}\right)^{1/3} \omega_p$

b. Growth rate  $\gamma \propto n_b^{1/3}$

## E. Buneman Instability

1a. Background of cold ions at rest

b. Cold "beam" of electrons moving through ions.  $n_e = n_i = n_0$ 

2. Dispersion Relation:  $D(k, \omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(kV - \omega)^2} = 0$

3. Yields  $\omega_r = \frac{1}{2} \left(\frac{m_e}{2m_i}\right)^{1/3} \omega_{pe}$  and  $\gamma = \frac{\sqrt{3}}{2} \left(\frac{m_e}{2m_i}\right)^{1/3} \omega_{pe}$

This fast growing instability occurs when cold electrons stream through cold ions.