PHYS:4761 Mathematical Methods of Physics I

Lecture #1: Infinite Series, Series of Functions, Binomial Theorem

I. Infinite Series

A. Fundamentals:

1. Infinite series are an extremely powerful way to represent functions, enabling solution by relatively straightforward means.
   a. Determine whether a series is convergent is often critical.

2. Infinite series $u_1 + u_2 + u_3 + \dots$

3. Def: Partial Sum $s_i \equiv \sum_{n=1}^{i} u_n$

4. Def: Convergence $\lim_{i \to \infty} s_i = S$

   then the infinite series converges $\sum_{n=1}^{\infty} u_n \equiv S$

   a. Necessary condition for convergence $\lim_{n\to\infty} u_n = 0$ (but not sufficient)

5. Def: Divergence $\lim_{i\to\infty} S_i = \pm\infty$

6. Def: Oscillatory: Ex: $\sum_{n=1}^{\infty} u_n = 1-1+1-1+\dots+(-1)^n+\dots$
   often classed with divergent. (not convergent)

B. Two Important Series:

1. Geometric Series a. $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$

   NOTE! Lower limit $n=0$.

   b. Partial Sum $S_n = \frac{1-r^n}{1-r}$

   c. Sum: $\lim_{n\to\infty} S_n = \boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}}$, convergent for $|r|<1$
   divergent for $r \geq 1$, $r < -1$
   oscillatory for $r = -1$

2. Harmonic Series a. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

   b. Although $\lim_{n\to\infty} \frac{1}{n} \Rightarrow 0$, it is not sufficient for convergence

   c. The Harmonic Series diverges $\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$

Both of these series are valuable for the Comparison Test.

C. The Comparison Test for Convergence

1. Convergence: If $0 \leq u_n \leq a_n$ for all n, and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} u_n$ is convergent.

2. Divergence: If $0 \leq b_n \leq v_n$ for all n, and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} v_n$ diverges.

D. D'Alembert Ratio Test (Easy Convergence Test to Apply)

1. If $\frac{a_{n+1}}{a_n} \leq r < 1$ for $n > N$ and r is independent of n, $\sum_{n=1}^{\infty} a_n$ is convergent

I. D. (Continued)

2. If $\frac{a_{n+1}}{a_n} \geq 1$ for $n > N$, then $\sum_{n=1}^{\infty} a_n$ is divergent

3. Simple Limiting Form $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \begin{cases} <1 & \text{convergence} \\ >1 & \text{divergence} \\ =1 & \text{indeterminate} \end{cases}$

4. If indeterminate, a more sensitive test is necessary:
   For example: a) Cauchy Root Test
   b) Kummer's Theorem
   c) Gauss's Test ← Good choice when D'Alembert Ratio Test is indeterminate.

5. Ex: Harmonic Series (Failure of D'Alembert Ratio Test)

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1$$

<u>BUT</u> cannot choose an $r < 1$ independent of n, since $\lim_{n\to\infty} \frac{n}{n+1} = 1$.
$\Rightarrow$ Test indeterminate!

E. <u>Cauchy Integral Test</u>:

1. Let $f(x)$ be a continuous, monotonically decreasing function in which $f(n) = a_n$.

Then $\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges if } \int_1^{\infty} f(x)\,dx \text{ is finite.} \\ \text{diverges if } \int_1^{\infty} f(x)\,dx \text{ is infinite.} \end{cases}$

F. <u>Alternating Series</u>

1. Generally convergence is more rapid due to cancellation.

2. <u>Leibniz Criterion</u> (Strict sign alternation)
For $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ with $a_n > 0$, if a_n is monotonically decreasing <u>and</u> $\lim_{n\to\infty} a_n = 0$, the series converges.

3. Absolute Convergence

a. A series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |u_n|$ converges.

b. Otherwise, the series is termed conditionally convergent.

c. Ex: Alternating Harmonic Series:
i) $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1}$ ⇒ convergent by Leibniz Criterion
ii) But $\sum_{n=1}^{\infty} |(-1)^{n-1} n^{-1}| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
iii) Thus, this series is conditionally convergent.

4. Conditions for Operating on Series

a. An absolutely convergence series may be reordered
i) Series sum is independent of order in which terms are added.
ii) You may add, subtract, or multiply termwise two absolutely convergent series; the resulting series is absolutely convergent.
iii) You may multiply whole series: the limit of the product is simply the product of the limits.

b. Riemann's Theorem: By a rearrangement of terms, a conditionally convergent series may be made to convergence to any desired value or to diverge.

BOTTOM LINE: Be careful with conditionally convergent series.

G. Improvement of Convergence

1. The rate of convergence may be improved by forming a linear combination of a slowly converging series with a known series.
2. Important for efficient numerical evaluation of series.

II. Series of Functions

A. **Basics**: 1. Consider each term is a function, $u_n = u_n(x)$

2. Partial Sum $\quad S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$

3. Series sum:
$$\sum_{n=1}^{\infty} u_n(x) = S(x) = \lim_{n\to\infty} S_n(x)$$

B. Uniform Convergence:

1. If for any small $\varepsilon > 0$, there exists a number N, independent of x over interval $[a, b]$ (that is, $a \le x \le b$) such that
$$|S(x) - S_n(x)| < \varepsilon \quad \text{for all } n > N,$$
then the series is uniformly convergent in $[a, b]$.

2. Note that absolute and uniform convergence are different concepts.

3. Weierstrass M (Majorant) Test (for uniform convergence)

a. A series $\sum_{n=1}^{\infty} u_n(x)$ will be uniformly convergent in $[a, b]$

if we can construct a convergent series $\sum_{n=1}^{\infty} M_n$ where $M_n \ge |u_n(x)|$ for all x in $[a, b]$.

b. NOTE: This test requires absolute convergence to establish uniform convergence.

C. Properties of Uniformly Convergent Series

1. If $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent in $[a, b]$ and $u_n(x)$ are continuous:

a. The Sum $S(x) = \sum_{n=1}^{\infty} u_n(x)$ is continuous

Almost always satisfied → b. Sum of integrals is equal to integral of sum: $\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx$

More restrictive → c. Sum of derivatives is equal to derivative of sum:
$$\frac{d}{dx} S(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x) \quad \text{if } \frac{du_n(x)}{dx} \text{ is continuous in } [a,b];\ \sum_{n=1}^{\infty} \frac{du_n(x)}{dx} \text{ is uniformly convergent.}$$

D. Taylor's Expansion

1. Perhaps one of the physicist's most widely used tools is the Taylor Expansion of a function into a power series.

2a. Assuming a function $f(x)$ has n continuous derivatives in $[a,b]$, one may integrate $f^{(n)}(x)$ n times to obtain:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \underbrace{R_n}_{\text{Remainder}}$$

b. This is an exact expression with remainder

$$R_n = \int_a^x dx_n \dots \int_a^{x_2} dx_1 \, f^{(n)}(x_1)$$

c. The mean value theorem, $\int_a^x g(x)dx = (x-a)g(\xi)$ for $a \le \xi \le x$, can be used to estimate R_n.

$$\Rightarrow \quad R_n = \frac{(x-a)^n}{n!} f^{(n)}(\xi)$$

d. NOTE: In this form, the series is not infinite, and therefore converges. The only question is the magnitude of R_n.

3. Taylor's Series: When $\lim_{n\to\infty} R_n = 0$,

$$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a)}$$

where $0! \equiv 1$.

This is the value at point x in terms of value and derivatives at reference point a

4. Maclaurin Series: Set $a=0$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

II. D. (Continued)

5. Ex: Power Series representation of $f(x) = e^x$

a. Note: $f'(x) = e^x$, so $f^{(n)}(0) = 1$.

b. Thus, using Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

c. Check convergence by D'Alembert Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} x = \lim_{n\to\infty} \frac{x}{n+1} = 0!$$ Convergent for $-\infty < x < \infty$!

d. Absolute value of series also converges $\Rightarrow$ Absolutely convergent!

E. Properties of Power Series

1. General Form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n$

2. If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = R^{-1}$, Series converges for $-R < x < R$ with a Radius of Convergence R.

3. a. Power series are uniformly and absolutely convergent for any interior interval $-S \le x \le S$ where $S < R$.
[Can be proven by Weierstrass M test].

b. Since $u_n(x) = a_n x^n$ are all continuous & $f(x)$ is uniformly convergent,
$f(x)$ must be continuous in $-S \le x \le S$.

4. a) Thus, infinite power series can only represent continuous functions!

b. Discontinuous functions (sawtooth wave, square wave) are often expressed as infinite series of trigonometric functions (sine, cosine $\Rightarrow$ Fourier Transform).

5. Uniqueness Theorem: Any power series representation is unique.

a. Proof: i) $\sum_{n=0}^{\infty} a_n x^n \stackrel{?}{=} \sum_{n=0}^{\infty} b_n x^n$ ii) Set $x=0 \Rightarrow a_0 = b_0$

iii) Differentiate & set $x=0 \Rightarrow a_1 = b_1$, etc ... $a_n = b_n$!

b. This property is extremely valuable in solving physics problems.

6. l'Hôpital's Rule: If the ratio of two differentiable functions $f(x)$ and $g(x)$ becomes indeterminate $\left(\frac{0}{0}\right)$ at $x = x_0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

a. Ex: Use power series to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$

i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

ii) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} = 1$

III Binomial Theorem:

A. Binomial Expansion

1. Applying the Maclaurin expansion to $(1+x)^m$ where m need not be positive nor integral.

$$\boxed{(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots}$$

2. Binomial Coefficients: a. In general, $\binom{m}{n} = \frac{m(m-1)\ldots(m-n+1)}{n!}$,

so $(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$

b. For an integer $m>0$, $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ "m choose n" Number of ways to choose n out of m objects.

3. Ex: Relativistic Energy: $E = mc^2\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

a. $E = \underbrace{mc^2}_{\text{Rest Energy}} + \underbrace{\frac{1}{2}mv^2}_{} + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \ldots$

↳ Classical limit of Kinetic energy for $v \ll c$.