Lecture #10 Tensor Analysis

I. Tensors

A. Introduction

1. Tensors arise in a number of important areas in physics: general relativity, electrodynamics, stress and strain, moment of inertia.
2. Scalars – tensors of rank 0
   Vectors – tensors of rank 1
3. Def: A tensor of rank n in a d-dimensional space:
   a. Components have n indices, each running 1 to d. Total of d^n components
   b. Components of tensor transform in a specified manner under coordinate transformations.

B. Covariant and Contravariant Tensors

1. For a rotational transformation from $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ to $(\hat{e}_1', \hat{e}_2', \hat{e}_3')$

$$(A_i)' = \sum_j (\hat{e}_i' \cdot \hat{e}_j) A_j = \underbrace{\sum_j \left(\frac{\partial x_i'}{\partial x_j}\right) A_j}_{\text{chain rule to convert } A_j \text{ into } A_i'}$$

2. But the gradient of a scalar ϕ transforms slightly differently.
   a. For $(\nabla\phi)_j = \left(\frac{\partial \phi}{\partial x_j}\right)\hat{e}_j$, $\quad (\nabla\phi)_i' = \frac{\partial \phi}{\partial x_i'} = \sum \left(\frac{\partial x_j}{\partial x_i'}\right)\frac{\partial \phi}{\partial x_j}$
3. NOTE: There is a subtle difference in these transformations.
   a. $\left(\frac{\partial x_i'}{\partial x_j}\right)_{x_k} \qquad \left(\frac{\partial x_j}{\partial x_i'}\right)_{x_k'}$ Different quantities are held fixed!
   b. In Cartesian coordinates, these are the same!
   c. But, in non-Cartesian systems, these transformations generally differ.

I. B. (Continued)

4. Contravariant vs Covariant Vectors:

a. Contravariant: $$(A')^i = \sum_j \frac{\partial (x')^i}{\partial x^j} A^j$$ superscript

b. Covariant $$(A')_i = \sum_j \frac{\partial x^j}{\partial (x')^i} A_j$$ subscript

5. Einstein Convention for Summation Notation

a. Unsummed index above (i) appears in same position on both sides.

⇒ NOTE: A superscript in the denominator is treated as a subscript.

b. The summed index (j) occurs once as upper, once as lower.

c. Einstein Convention: Omit summation sign. A repeated index appearing as an upper & lower index is assumed to be summed.

$$(A')^i = \frac{\partial (x')^i}{\partial x^j} A^j \quad \leftarrow \text{upper} \quad \leftarrow \text{lower} \quad \Rightarrow \text{implicit} \sum_{j=1}^{d}$$

C. Tensors of Rank 2 (3×3 Matrices)

1. When dealing with tensors, summation notation is often easier to manage than matrix multiplication notation.

2. Different Types:

a. Contravariant $$(A')^{ij} = \frac{\partial (x')^i}{\partial x^k} \frac{\partial (x')^j}{\partial x^l} A^{kl}$$ k, l repeated $\Rightarrow \sum_{k,l}$

b. Mixed $$(B')^i_j = \frac{\partial (x')^i}{\partial x^k} \frac{\partial x^l}{\partial (x')^j} B^k_l$$

c. Covariant $$(C)_{ij} = \frac{\partial x^k}{\partial (x')^i} \frac{\partial x^l}{\partial (x')^j} C_{kl}$$

3. Invariance under transformations is used to express universal physical laws $\Rightarrow$ This makes tensor analysis important.

4. Matrix Form

a. $$\underset{\approx}{A} = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix}$$

b. Similarity transformation

$$(A')^{ij} = \sum_{kl} S_{ik} A^{kl} (S^T)_{lj} \Leftrightarrow \underset{\approx}{A}' = \underset{\approx}{S}\underset{\approx}{A}\underset{\approx}{S}^T$$

D. Tensor Properties and Operations

1. Addition as expected for matrices, $A^{ij} + B^{ij} = C^{ij}$

2. Symmetry: a. Symmetric if $A^{mn} = A^{nm}$

b. Anti-symmetric if $A^{mn} = -A^{nm}$

c. Decomposition:

$$A^{mn} = \underbrace{\tfrac{1}{2}(A^{mn} + A^{nm})}_{\text{Symmetric}} + \underbrace{\tfrac{1}{2}(A^{mn} - A^{nm})}_{\text{Anti-symmetric}}$$

3. Kronecker Delta: a. Mixed rank 2 tensor, δ^k_l

b. Does it transform properly? $$(\delta')^i_j = \frac{\partial (x')^i}{\partial x^k} \frac{\partial x^l}{\partial (x')^j} \delta^k_l = \frac{\partial (x')^i}{\partial x^k} \frac{\partial x^k}{\partial (x')^j} = \frac{\partial (x')^i}{\partial (x')^j} = (\delta')^i_j$$

c. Isotropic: δ_l^k has same components in all rotated coordinates.

4. Contraction: a. For Vectors, $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_i A_i B_i$

b. For tensor, set two indices equal to each other, B_i^i (upper and lower), and sum according to summation convention

c. Ex: $$(B')^i_i = \underbrace{\frac{\partial (x')^i}{\partial x^k} \frac{\partial x^l}{\partial (x')^i}}_{=\delta^l_k} B^k_l = \frac{\partial x^l}{\partial x^k} B^k_l = \delta^l_k B^k_l = B^k_k$$

d. Contraction Reduces a tensor rank by 2 $\Rightarrow$ Rank 2 Tensor $\rightarrow$ Scalar.

e. NOTE: In Matrix Analysis, $$B^k_k \Rightarrow \sum_k B^k_k = B^1_1 + B^2_2 + B^3_3 = \text{Trace}(\underset{\approx}{B})$$

I. D. (Continued)

5. Direct Product: (Sum ranks of tensors)

a. $C^{ij}_{k\ell m} = A^i_k B^j_{\ell m}$ — Simply multiply, component by component.

Rank: $5 = 2 + 3$ — Easy in summation notation.

b. Contravariance or covariance of each index must be maintained.

6. Inverse Transformation:

a. $(A')^j = \frac{\partial (x')^j}{\partial x^i} A^i \xrightarrow{\text{inverse}} A^i = \frac{\partial x^i}{\partial (x')^j} (A')^j$

b. But $\Rightarrow$ x^k fixed $\qquad (x')^k$ fixed

c. Check Inverse: $(A')^k = \frac{\partial (x')^k}{\partial x^i}\left[A^i\right] = \frac{\partial (x')^k}{\partial x^i}\left[\frac{\partial x^i}{\partial (x')^j}(A')^j\right] = \frac{\partial (x')^k}{\partial (x')^j}(A')^j = \delta^k_j (A')^j = (A')^k$

7. Quotient Rule: Used to establish tensor nature.

a. If A_{ij} & $B^{k\ell}$ are tensors, then direct product $C^{k\ell}_{ij} = A_{ij} B^{k\ell}$ is a tensor

b. What about inverse problem?

$$K_{k\ell}\, C^{k\ell}_{ij} = A_{ij} \qquad (K_{k\ell} \text{ is inverse of } B^{k\ell})$$

c. Quotient Rule: If the equation of interest holds in all coordinate systems, then $\underset{\approx}{K}$ is a tensor of indicated rank & contra/co-variant character.

E. Pseudotensors:

1. For vectors, $\quad \underset{\sim}{A}' = \underset{\approx}{S}\underset{\sim}{A}$ $\qquad$ vector

$\underset{\sim}{A}' = \det(\underset{\approx}{S})\, \underset{\approx}{S}\, \underset{\sim}{A}$ $\qquad$ pseudovector

2. Pseudotensors are the extension of this concept.

a. Require additional sign factor in transformation rule.

I. E (Continued)

3. <u>Quotient Rule and psuedotensors:</u> $\underset{\approx}{T} \rightarrow$ tensor

$\underset{\approx}{P} \rightarrow$ psuedotensor

a. $\underset{\approx}{T} \otimes \underset{\approx}{T} = \underset{\approx}{T}$ b. $\underset{\approx}{P} \otimes \underset{\approx}{T} = \underset{\approx}{P}$

$\underset{\approx}{P} \otimes \underset{\approx}{P} = \underset{\approx}{T}$ $\underset{\approx}{T} \otimes \underset{\approx}{P} = \underset{\approx}{P}$

4. <u>Ex: Levi-Civita Symbol</u> ε_{ijk}

a. ε_{ijk} is a <u>rank-3 psuedotensor</u> and is <u>isotropic</u>

II. Tensors in General Coordinates

A. The Metric Tensor

1. For non-Cartesian coordinate systems, we want to develop a systematic way of handling these more general <u>metric spaces</u>.

2. Consider a general (3D) coordinate system (q_1, q_2, q_3)

a. <u>Covariant basis vectors</u> $\underset{\sim}{\varepsilon}_i$

$$\underset{\sim}{\varepsilon}_i = \frac{\partial x}{\partial q_i}\underset{\sim}{\hat{e}}_x + \frac{\partial y}{\partial q_i}\underset{\sim}{\hat{e}}_y + \frac{\partial z}{\partial q_i}\underset{\sim}{\hat{e}}_z$$

b. <u>Arbitrary Vector:</u> $\underset{\sim}{A} = A^1\underset{\sim}{\varepsilon}_1 + A^2\underset{\sim}{\varepsilon}_2 + A^3\underset{\sim}{e}_3$

c. <u>NOTE:</u> i. $\underset{\sim}{A}$ is fixed object (unchanged by choice of coordinates)

ii. A^i coefficients form a contravariant vector

iii. $\underset{\sim}{\varepsilon}_i$ is a covariant basis vector.

iv. When coordinate system is changed, A^i and $\underset{\sim}{\varepsilon}_i$ change in complementary ways to yield a fixed $\underset{\sim}{A}$.

II A. (Continued)

3. $$(ds)^2 = \sum_{ij} (\underset{\sim}{\varepsilon}_i \, dq^i)(\underset{\sim}{\varepsilon}_j \, dq^j) = \underbrace{g_{ij} \, dq^i dq^j}_{\text{summation convention.}}$$

4. <u>Covariant Metric Tensor</u>: a. $$\boxed{g_{ij} = \underset{\sim}{\varepsilon}_i \cdot \underset{\sim}{\varepsilon}_j}$$

b. Since $(ds)^2$ is a scalar, <u>quotient rule</u> tells us that g_{ij} must be a <u>rank 2 covariant tensor</u>.

5. <u>Raising and Lowering Operations</u>

a. <u>Define</u>: <u>Contravariant Metric Tensor</u> g^{ij} by $g^{ik} g_{kj} = \delta^i_j$

b. g^{ij} is the <u>inverse</u> of g_{ij}.

c. Convert contravariant to covariant tensor $\quad g_{ij} F^j = F_i \quad$ "Lowering"

d. Convert covariant to contravariant tensor $\quad g^{ij} F_j = F^i \quad$ "Raising"

6. <u>Ex</u>: a. $$\underset{\sim}{A} = A^i \underset{\sim}{\varepsilon}_i = A^i [\delta_i^k] \underset{\sim}{\varepsilon}_k = A^i [g_{ij} g^{jk}] \underset{\sim}{\varepsilon}_k = A_j \underset{\sim}{\varepsilon}^j$$

b. Thus, the same vector can be represented in either contravariant or covariant bases.

7. To wrap up, we provide the following definitions.

a. <u>Contravariant basis vectors</u> $$\underset{\sim}{\varepsilon}^i = \frac{\partial q^i}{\partial x} \hat{\underset{\sim}{e}}_x + \frac{\partial q^i}{\partial y} \hat{\underset{\sim}{e}}_y + \frac{\partial q^i}{\partial z} \hat{\underset{\sim}{e}}_z$$

b. <u>Contravariant metric tensor</u> $$\boxed{g^{ij} = \underset{\sim}{\varepsilon}^i \cdot \underset{\sim}{\varepsilon}^j}$$

II. A. (Continued)

8. Ex: Spherical Polar Metric Tensor: $(q^1, q^2, q^3) = (r, \theta, \phi)$

a. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

b. Covariant Basis Vectors:
$$\underset{\sim}{\varepsilon}_r = \sin\theta\cos\phi\,\hat{\underset{\sim}{e}}_x + \sin\theta\sin\phi\,\hat{\underset{\sim}{e}}_y + \cos\theta\,\hat{\underset{\sim}{e}}_z$$
$$\underset{\sim}{\varepsilon}_\theta = r\cos\theta\cos\phi\,\hat{\underset{\sim}{e}}_x + r\cos\theta\sin\phi\,\hat{\underset{\sim}{e}}_y - r\sin\theta\,\hat{\underset{\sim}{e}}_z$$
$$\underset{\sim}{\varepsilon}_\phi = -r\sin\theta\sin\phi\,\hat{\underset{\sim}{e}}_x + r\sin\theta\cos\phi\,\hat{\underset{\sim}{e}}_y$$

c. Can also compute ~~compute~~ Contravariant basis Vectors from
$$r^2 \equiv x^2 + y^2 + z^2, \quad \cos\theta = \frac{z}{r}, \quad \tan\phi = \frac{y}{x}$$

d. Covariant Metric Tensor
$$g_{11} = \underset{\sim}{\varepsilon}_r \cdot \underset{\sim}{\varepsilon}_r = 1$$
$$g_{22} = \underset{\sim}{\varepsilon}_\theta \cdot \underset{\sim}{\varepsilon}_\theta = r^2$$
$$g_{33} = \underset{\sim}{\varepsilon}_\phi \cdot \underset{\sim}{\varepsilon}_\phi = r^2\sin^2\theta$$
$$\Rightarrow (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

e. Similarly Contravariant metric tensor
$$(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (r\sin\theta)^{-2} \end{pmatrix}$$

9. Ex: Minkowski Special Relativity Metric (4-vectors)

$$(g_{ij}) = (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{array}{l} \leftarrow \text{time} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{3D space} \end{array}$$

B. Covariant Derivatives

1. Much more complicated because $\underset{\sim}{\varepsilon}_i$ are, in general, not constant!

2a. Begin with transformation law: $(V')^i = \dfrac{\partial x^i}{\partial q^k} V^k$

b. Differentiate:
$$\frac{\partial (V')^i}{\partial q^j} = \frac{\partial x^i}{\partial q^k}\frac{\partial V^k}{\partial q^j} + \frac{\partial^2 x^i}{\partial q^j \partial q^k} V^k$$

III. B2 (Continued)

c. Write as a vector equation in x^i coordinates: $\underset{\sim}{\varepsilon}_k = \frac{\partial x^i}{\partial q^k}$, so

$$\underset{\sim}{V}' = V_x'\underset{\sim}{\hat{e}}_x + V_y'\underset{\sim}{\hat{e}}_y + V_z'\underset{\sim}{\hat{e}}_z \qquad \frac{\partial \underset{\sim}{V}'}{\partial q^j} = \frac{\partial V^k}{\partial q^j}\underset{\sim}{\varepsilon}_k + V^k \frac{\partial \underset{\sim}{\varepsilon}_k}{\partial q^j}$$

d. NOTE: i. $\frac{\partial \underset{\sim}{\varepsilon}_k}{\partial q^j}$ is a vector in the space spanned by $\underset{\sim}{\varepsilon}_i$.

ii. Thus, $\frac{\partial \underset{\sim}{\varepsilon}_k}{\partial q^j} = \Gamma^{\mu}_{jk}\underset{\sim}{\varepsilon}_\mu$ Christoffel symbol of second kind

3. Christoffel Symbol: a. $\Gamma^{m}_{jk} = \underset{\sim}{\varepsilon}^{m} \cdot \frac{\partial \underset{\sim}{\varepsilon}_k}{\partial q^j}$

b. NOTE: $\Gamma^{m}_{kj} = \Gamma^{m}_{jk}$

4. Thus, $\frac{\partial \underset{\sim}{V}'}{\partial q^j} = \frac{\partial V^k}{\partial q^j}\underset{\sim}{\varepsilon}_k + V^k \Gamma^{\mu}_{jk}\underset{\sim}{\varepsilon}_\mu$

5. Covariant Derivative: a. $\boxed{\frac{\partial \underset{\sim}{V}'}{\partial q^j} = \left(\frac{\partial V^k}{\partial q^j} + V^\mu \Gamma^{k}_{j\mu}\right)\underset{\sim}{\varepsilon}_k}$

b. Def: $\boxed{V^k_{;j} \equiv \frac{\partial V^k}{\partial q^j} + V^\mu \Gamma^{k}_{j\mu}}$

c. Mixed, Second Rank Tensor: i. Only combination $V^k_{;j}$ transforms as a tensor.

ii. $\frac{\partial V^k}{\partial q^j}$ and $V^\mu \Gamma^{k}_{j\mu}$ do not individually transform as tensors.

d. Includes changes in basis vectors $\underset{\sim}{\varepsilon}_k$ as q^j changes (dq^j) to determine derivative.