Lecture #11 Jacobians and Differential Forms

I. Jacobians

1. Jacobians tell us how multidimensional integrals transform under a change of coordinate systems: $dx\,dy\,dz \Rightarrow \underbrace{r^2\sin\theta}_{\text{Jacobian}}\,dr\,d\theta\,d\phi$

A. General Form

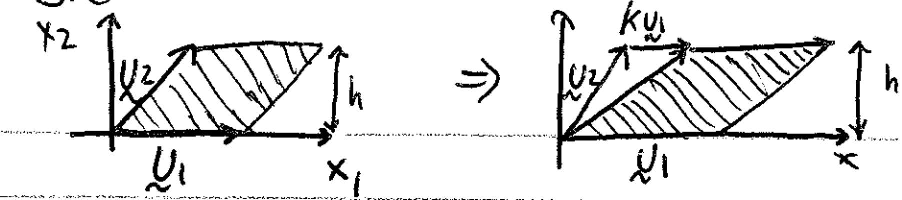
1. Consider changing from (x_1, x_2, x_3) to (u_1, u_2, u_3)
   a. Replace differential $dx_1 dx_2 dx_3$ with $\underset{\swarrow \text{Jacobian}}{J}\,du_1 du_2 du_3$
   b. $d\tau = J\,du_1 du_2 du_3$ is a "volume" of length du_1, du_2, du_3 in x_1, x_2, x_3 space.

2. Compute displacement (in x_i-space) corresponding to change in u_i.
   a. $$\underset{\sim}{ds}(u_1) = \left[\left(\frac{\partial x_1}{\partial u_1}\right)\underset{\sim}{\hat{e}_1} + \left(\frac{\partial x_2}{\partial u_1}\right)\underset{\sim}{\hat{e}_2} + \left(\frac{\partial x_3}{\partial u_1}\right)\underset{\sim}{\hat{e}_3}\right] du_1$$
   in (x_1, x_2, x_3) space
   $$\underset{\sim}{ds}(u_2) = \left[\left(\frac{\partial x}{\partial u_2}\right)\underset{\sim}{\hat{e}_1} + \left(\frac{\partial y}{\partial u_2}\right)\underset{\sim}{\hat{e}_2} + \left(\frac{\partial z}{\partial u_2}\right)\underset{\sim}{\hat{e}_3}\right] du_2, \text{ etc.}$$
   b. NOTE: Partial derivatives $\left(\frac{\partial x_i}{\partial u_j}\right)_{u_k}$ evaluated with $u_k = \text{const}$ $(k \neq j)$.

3. In 2D, "Area" would be $|\underset{\sim}{ds}(u_1)|$ times perpendicular component of $\underset{\sim}{ds}(u_2)$.

4. Matrix Form
$$\begin{pmatrix} \frac{ds(u_1)}{du_1} \\ \frac{ds(u_2)}{du_2} \\ \frac{ds(u_3)}{du_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{pmatrix} \begin{pmatrix} \underset{\sim}{\hat{e}_1} \\ \underset{\sim}{\hat{e}_2} \\ \underset{\sim}{\hat{e}_3} \end{pmatrix}$$

5. Jacobian can be computed by making the matrix above diagonal by adding multiples of one row to other rows
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$
   a. NOTE: This process does not change "volume" (or value of determinant.)

b. 

"Area" $= u_1 h$ is unchanged!

c. "Volume" of Jacobian is just product of diagonal elements.

d. Since this process does not change the value of the determinant, it is the determinant that is the Jacobian.

6.

$$\boxed{d\tau = J\, du_1\, du_2\, du_3}, \quad \boxed{J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_3} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix} \equiv \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)}}$$

7. Ex: 2D Jacobian

a. $(x, y) \Rightarrow (u, v)$ $\quad dx\,dy = J\,du\,dv$

where $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) - \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)$.

B. Inverse of Jacobian

1. If $dx_1 dx_2 dx_3 = J\, du_1 du_2 du_3$, then $du_1 du_2 du_3 = (J^{-1})\, dx_1 dx_2 dx_3$.

2. We could use $u_1 = u_1(x_1, x_2, x_3)$, $u_2 = u_2(x_1, x_2, x_3)$, ... to compute $J^{-1} = \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$

3. It is sometimes much easier to compute J^{-1} than J.

4. Ex: Spherical Coordinates:

a. Consider transformation from (r, θ, ϕ) to (x, y, z) $\Rightarrow J = \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)}$

where $r = \sqrt{x^2 + y^2 + z^2}$ $\quad \theta = \cos^{-1}\left(\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right)$ $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

b. But $J^{-1} = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ is the inverse where

$x = r \sin\theta \cos\phi$ $\quad y = r \sin\theta \sin\phi$ $\quad z = r\cos\theta$

I.B.4. (Continued)

c. So, $J^{-1} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$

$= r^2\sin^3\theta\sin^2\phi + r^2\cos^2\theta\sin\theta\cos^2\phi + r^2\cos^2\theta\sin\theta\sin^2\phi + r^2\sin^3\theta\cos^2\phi$

$= r^2\sin\theta\left[\sin^2\theta + \cos^2\theta\right] = r^2\sin\theta$

d. Thus $\boxed{dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\phi} \Rightarrow \boxed{dr\,d\theta\,d\phi = \frac{1}{r^2\sin\theta}dx\,dy\,dz}$

II. Differential Forms

A. Introduction

1. Differential forms are a powerful generalization of vector and tensor analysis in curvilinear coordinate systems.

2. Enables treatment of topology (connectivity of spaces) in physics.

3. Basic Quantities: Differentials dx, dy, dz

a. Associated with linearly independent directions in space (orthogonal)

4. 1-forms $\omega = A(x,y,z)dx + B(x,y,z)\,dy + C(x,y,z)dz$

a. Use as integrand in line integrals

5. 2-forms: $\omega = F(x,y,z)dx\wedge dy + G(x,y,z)dx\wedge dz + H(x,y,z)\,dy\wedge dz$

a. Use as integrand of surface integrals

6. 3-forms: $\omega = K(x,y,z)\,dx\wedge dy\wedge dz$

a. Use as integrand of volume integrals

7. Exterior Algebra (Grassman Algebra)

a. "Wedge" $\wedge$

b. Permutational Anti-symmetry

c. Properties

$$\boxed{\begin{aligned}(a\omega_1 + b\omega_2)\wedge\omega_3 &= a\,\omega_1\wedge\omega_3 + b\,\omega_2\wedge\omega_3\\ (\omega_1\wedge\omega_2)\wedge\omega_3 &= \omega_1\wedge(\omega_2\wedge\omega_3)\\ a(\omega_1\wedge\omega_2) &= (a\omega_1)\wedge\omega_2\\ dx_i\wedge dx_j &= -dx_j\wedge dx_i\end{aligned}}$$

d. In general, any differential form can be reduced to a function multiplying a differential dx_i or wedge product $dx_i \wedge dx_{x_2}$.

$$(a\,dx_1)\wedge(b\,dx_2) = ab(dx_1\wedge dx_2) = -ab(dx_2\wedge dx_1)$$

e. By antisymmetry, $\boxed{dx_i \wedge dx_i = 0}$

8. p-forms: a. A form of order p (contains p differentials)

b. Ordinary function or scalar is a 0-form.

c. Order p of form cannot be larger than dimension of space, $\boxed{p \le d}$

9. Ex: $\omega = (3dx + 4dy - dz)\wedge(dx - dy + 2dz)$

$= -3dx\wedge dy + 6dx\wedge dz + 4dy\wedge dx + 8dy\wedge dz - dz\wedge dx + dz\wedge dy$

$= -7dx\wedge dy + 7dy\wedge dz - 7dz\wedge dx = \boxed{7(-dx\wedge dy + dy\wedge dz - dz\wedge dx)}$

B. Complementary Differential Forms

1. Complementary (or dual) form includes differentials not included in original form.

a. In dimension d, dual to p-form is a $(d-p)$-form.

2. Def: Hodge Star Operator, $*\omega$

a. For $*\omega$, where ω is a p-form, first form wedge product of $(d-p)$ differentials not in ω.

b. Sign of $*\omega$ is determined by permutations needed to bring (indices of ω) followed by (indices of ω') to standard order.

II. B. 2. (Continued)

c. Formally, to find the complementary form requires the specification of the metric and selection of a reference order.

d. $*\omega$ also includes $(-1)^{\mu}$, where μ is number of differentials in ω whose metric tensor diagonal element is -1.

3. Ex: Complementary Forms in $\mathbb{R}^3$

$$*1 = dx_1 \wedge dx_2 \wedge dx_3$$
$$*dx_1 = dx_2 \wedge dx_3, \quad *dx_2 = dx_3 \wedge dx_1, \text{ etc.}$$
$$*(dx_1 \wedge dx_2) = dx_3, \quad *(dx_3 \wedge dx_1) = dx_2, \text{ etc.}$$
$$*(dx_1 \wedge dx_2 \wedge dx_3) = 1$$

4. Ex: Minkowski Space (dt, dx_1, dx_2, dx_3)

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

a. $\boxed{*1 = dt \wedge dx_1 \wedge dx_2 \wedge dx_3}$

NOTE: 1 has no differentials, so $\mu = 0 \Rightarrow (-1)^{\mu} = 1$.

b. $\boxed{*(dt \wedge dx_1 \wedge dx_2 \wedge dx_3) = -1}$

NOTE: Here $\mu = 3$, so $(-1)^{\mu} = -1$!

c. $\boxed{*(dt \wedge dx_1) = -dx_2 \wedge dx_3}$

NOTE: $\mu = 1$, so $(-1)^{\mu} = -1$

5. Connection to Vector Analysis

a. $A = A_x dx + A_y dy + A_z dz, \quad B = B_x dx + B_y dy + B_z dz$

b. $*(A \wedge B) = (A_y B_z - A_z B_y) dx + (A_z B_x - A_x B_z) dy + (A_x B_y - A_y B_x) dz$

$$\Rightarrow *(A \wedge B) \Leftrightarrow \underset{\sim}{A} \times \underset{\sim}{B}$$

c. $$*(A \wedge B \wedge C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C})$$

d. Generalizable to arbitrary dimension and metric.

III. (Continued)

C. Differentiating Forms

1. Def: Exterior Derivative $d\omega$

a. For ω a p-form, ω' a p'-form, and f a function (0-form)

$$d(\omega+\omega') = d\omega + d\omega' \qquad (p=p')$$

$$d(f\omega) = (df)\wedge\omega + f\,d\omega$$

$$d(\omega\wedge\omega') = d\omega\wedge\omega' + (-1)^p \omega \wedge d\omega'$$

$$d(d\omega) = 0$$

$$df = \sum_{j=1}^{d} \frac{\partial f}{\partial x_j} dx_j$$

b. NOTE: The derivative of a p-form is a (p+1)-form.

2. Connection to Vector differential operators

a. d(0-form) ⇒ Gradient: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (\nabla f)_x dx + (\nabla f)_y dy + (\nabla f)_z dz$

b. d(1-form) ⇒ Curl:

$$d(A_x dx + A_y dy + A_z dz) = (\nabla\times\underset{\sim}{A})_x dy\wedge dz + (\nabla\times\underset{\sim}{A})_y dz\wedge dx + (\nabla\times\underset{\sim}{A})_z dx\wedge dy$$

c. d(2-form) ⇒ Divergence:

$$*d(B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy) = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla\cdot\underset{\sim}{B}$$

d. $d(df) = 0 \Rightarrow \nabla\times(\nabla f) = 0$ (Use $\underset{\sim}{A} = \nabla f$ in b) above)

e. $d[d(A_x dx + A_y dy + A_z dz)] = \nabla\cdot(\nabla\times\underset{\sim}{A})\, dx\wedge dy\wedge dz = 0$

(Use $\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$ in (c) above)

3. Ex: Maxwell's Equations in Minkowski Space (dt, dx, dy, dz)

a. $F = -E_x\, dt\wedge dx - E_y\, dt\wedge dy - E_z\, dt\wedge dz$
$\quad + B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy$ replaces $\underset{\sim}{E}$ & $\underset{\sim}{B}$

b. Sources: $J = \rho\, dx \wedge dy \wedge dz - J_x\, dt \wedge dy \wedge dz - J_y\, dt \wedge dz \wedge dx - J_z\, dt \wedge dx \wedge dy$

replaces ρ & $\underset{\sim}{J}$.

c. Use units with $\varepsilon_0 = \mu_0 = c = 1$.

d. Thus,
$$\boxed{dF = 0} \Rightarrow \nabla \times \underset{\sim}{E} + \frac{\partial \underset{\sim}{B}}{\partial t} = 0 \quad \text{and} \quad \nabla \cdot \underset{\sim}{B} = 0$$

e.
$$\boxed{d(*F) = 0} \Rightarrow \nabla \cdot \underset{\sim}{E} = \rho \quad \text{and} \quad \nabla \times \underset{\sim}{B} - \frac{\partial \underset{\sim}{E}}{\partial t} = \underset{\sim}{J}$$

f.
$$\boxed{dJ = 0} \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{J} = 0$$

D. Integrating Forms

1. Integral of a 1-form ω in 2D space over curve C
$$\int_C \omega = \int_C [A_x\, dx + A_y\, dy]$$

a. For a parametric form of C as $x(t)$ & $y(t)$ from t_P to t_Q
$$\int_C \omega = \int_{t_P}^{t_Q} \left[A_x(t)\frac{dx}{dt} + A_y(t)\frac{dy}{dt}\right] dt$$

b. For a conservative force, ω is exact, such that $\boxed{\omega = df(x,y)}$

c. Then $\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, so $\int_P^Q \omega = f(Q) - f(P)$ (independent of path)

2. Surface Integrals of 2-forms

a. $$\int_S \omega = \int_S B(x,y)\, dx \wedge dy$$ Here $\int dx \wedge dy \Rightarrow \int dx\, dy$

b. Change of variables and Jacobian:

i. $x = au + bv \Rightarrow dx = a\,du + b\,dv$
$y = eu + fv \Rightarrow dy = e\,du + f\,dv$

ii. $dx \wedge dy = (a\,du + b\,dv) \wedge (e\,du + f\,dv) = (af - be)\, du \wedge dv$

iii. But, $$af - be = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix} = J \Rightarrow \boxed{dx \wedge dy = J\, du \wedge dv}$$

naturally included!

C. NOTE! Since $dx \wedge dy = -dy \wedge dx$, the "area" in differential forms is oriented. Sign is necessary in Jacobian transformation.

3. Stoke's Theorem:

a. If R is a simply-connected region of a p-dimensional differentiable manifold in n-dimensional space $(n \geq p)$

and b. If R has boundary ∂R, of dimension $(p-1)$

and c. ω is a $(p-1)$-form defined on R and it boundary, with derivative $d\omega$

$$\boxed{\int_R d\omega = \int_{\partial R} \omega}$$

d. Applies for manifolds of any dimension (includes Stoke's & Gauss' Thm's from vector analysis).

4. Ex: Usual 3D case of Stoke's Theorem

a. $\omega = A_x dx + A_y dy + A_z dz$

b. $$d\omega = \underbrace{\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]}_{(\nabla \times \underset{\sim}{A})_x} dy \wedge dz + \underbrace{\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]}_{(\nabla \times \underset{\sim}{A})_y} dz \wedge dx + \underbrace{\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]}_{(\nabla \times \underset{\sim}{A})_z} dx \wedge dy$$

c. Reference Order $(dx, dy, dz) \Rightarrow dy \wedge dz \Rightarrow d\underset{\sim}{\sigma}_x$, $dz \wedge dx \Rightarrow d\underset{\sim}{\sigma}_y$

$$\int_C \omega = \int_C (A_x dx + A_y dy + A_z dz) = \underbrace{\int_C \underset{\sim}{A} \cdot d\underset{\sim}{r} = \int_S (\nabla \times \underset{\sim}{A}) \cdot d\underset{\sim}{\sigma}}_{\text{Usual Vector Analysis version of Stoke's Theorem}}$$