Lecture #12 Vectors in Function Spaces

I. Vector Spaces

A. Introduction

1. Vector spaces deal with expansions in a series of functions
   a. For example, polynomials, $a_n x^n$, or trigonometric functions $\sin(nx)$
2. An arbitrary function $f(x)$ can be expressed as an expansion in these basis functions.
3. The coefficients a_n transform similar to vector components, and operators can act on the functions (and associated components)

B. Vectors in Function Spaces

1. Extend concepts of vector analysis to more general situations.
2. Example of 2D Vector Space

a.

	Function Space	Vectors
Basis:	$\phi_1(s)$ & $\phi_2(s)$	$\hat{e}_1$ & $\hat{e}_2$
Function/Vector:	$f(s) = a_1\phi_1(s) + a_2\phi_2(s)$	$\tilde{A} = A_1\hat{e}_1 + A_2\hat{e}_2$
Coordinate in Space:	(a_1, a_2)	(A_1, A_2)

s in an independent variable (not space)

3. Def: Linear Vector Space: a. Basis $\phi_1(s)$, $\phi_2(s)$
   b. Set of function $f(s)$ can be built by linear combination of $\phi_1(s)$ & $\phi_2(s)$
   $\Rightarrow f(s) = a_1\phi_1(s) + a_2\phi_2(s)$
4. Properties:
   a. Addition: $g(s) = b_1\phi_1(s) + b_2\phi_2(s)$
   $h(s) = f(s) + g(s) = (a_1+b_1)\phi_1(s) + (a_2+b_2)\phi_2(s)$
   NOTE: Sum of two members is also a member of function space.
   b. Multiplication by Scalar: $u(s) = k f(s) = ka_1\phi_1(s) + ka_2\phi_2(s)$.
5. Vector space is closed under an operation if that operation always produces another member of the vector space.

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6. <u>Basis Functions</u>:

a. Can be functions, compound objects (Pauli matrices), or even some abstract quantity.

b. <u>Dimension</u> is number of basis functions: Can be small, large, infinite

c. Basis functions must be <u>linearly independent</u> ("orthogonal"), so that any function is represented by a <u>unique</u> linear combination.

7. <u>Examples</u>:

a. <u>Finite basis of dimension 3</u>: (Spanned by 3 functions)

i. $P_0(s)=1$, $P_1(s)=s$, $P_2(s)=\frac{3}{2}s^2-\frac{1}{2}$ (Legendre)

ii. Functions represented in terms of basis functions.

$$f_1(s)=s+3 = 3P_0(s)+P_1(s)$$
$$f_2(s)=s^2=\frac{2}{3}P_2(s)+\frac{1}{3}P_0(s)$$

iii. Any quadratic in s is a member of vector space, $c_0+c_1s+c_2s^2$

iv. Operations: $g(s)=2f_1(s)-f_2(s)=2\left[3P_0(s)+P_1(s)\right]-\left[\frac{2}{3}P_2(s)+\frac{1}{3}P_0(s)\right]$

NOTE: I do not need definitions $P_n(s)$ to add! $\longrightarrow$ $=\frac{17}{3}P_0(s)+2P_1(s)-\frac{2}{3}P_2(s)$

v. Any <u>linearly independent</u> basis will do: $\phi_0=1$, $\phi_1=s$, $\phi_2=s^2$

b. <u>Infinite polynomial basis</u>: $\phi_n(s)=s^n$ $n=0,1,2,\ldots$

i. Represents any function that can be represented by Maclaurin series

$$f(s)=\sum_{n=0}^{\infty}\frac{s^n}{n!}f^{(n)}(0)$$ (Cannot represent discontinuous functions)

where $a_n=\frac{f^{(n)}(0)}{n!}$

ii. Only valid over range in s for which series <u>converges</u>.

$\Rightarrow$ In physics, usually convergence occurs naturally.

c. <u>Electron Spin</u>: Two spin states α, β

i. $f=a_1\alpha+a_2\beta$ $g=b_1\alpha+b_2\beta$

ii. <u>Don't need to know definition of α,β to add</u>: $f+ig=(a_1+ib_1)\alpha+(a_2+ib_2)\beta$.

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C. Scalar Product:

1. Scalar product $\langle f|g\rangle$
   a. $\langle f|f\rangle$ is a scalar (Corresponds to $|\underset{\sim}{A}|^2$)
   b. Linear in both f and g.
2. Dirac Notation: bra-ket ⇒ "bracket"
   a. ket $|g\rangle$
   b. bra $\langle f|$
   c. When combined, interpreted as scalar product $\langle f|g\rangle$.
3. Scalar Product can have a wide range of definitions.
   a. Ex: $\langle f(s)|g(s)\rangle = \int_a^b f^*(s)\,g(s)\,W(s)\,ds$
   b. Limits a & b and weight function W(s) define scalar product.
   c. Since $\langle f|f\rangle \geq 0$ is like a "length", $W(s) \geq 0$ over $[a,b]$.

4. Alternatively, you may define scalar product in terms of its values.
   a. Ex. Electron Spin $\langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1$, $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle = 0$.
   b. Since $\langle f|g\rangle$ is linear in f & g;
   if $f = a_1\alpha + a_2\beta$ and $g = b_1\alpha + b_2\beta$, then
   $\langle f|g\rangle = a_1b_1\langle\alpha|\alpha\rangle + a_1b_2\cancel{\langle\alpha|\beta\rangle} + a_2b_1\cancel{\langle\beta|\alpha\rangle} + a_2b_2\langle\beta|\beta\rangle = a_1b_1 + a_2b_2$.

5. If basis functions are equivalent to orthogonal coordinate system then scalar product is equivalent to dot product.

D. Hilbert Space, $\mathcal{H}$

1. Def: Hilbert Space: A vector space closed under addition and scalar multiplication that has a defined scalar product for all pairs of members.
2. $\mathcal{H}$ is spanned by set of basis functions ϕ_i
   $\mathcal{H}$ is complete because every function in $\mathcal{H}$ is a linear form $f(s) = \sum_n a_n \phi_n(s)$.

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3. Scalar Product: a. $\langle f|f\rangle \geq 0$ Norm: $\|f\| \equiv \langle f|f\rangle^{\frac{1}{2}}$

b. $\langle g|f\rangle^* = \langle f|g\rangle$

c. $\langle f|g+h\rangle = \langle f|g\rangle + \langle f|h\rangle$

d. $k\langle f|g\rangle = \langle f|kg\rangle$, $\langle kf|g\rangle = k^*\langle f|g\rangle$

4. Example Hilbert Space: a. $P_0(s) = 1$, $P_1(s) = s$, $P_2(s) = \frac{3}{2}s^2 - \frac{1}{2}$

b. $\langle f|g\rangle \equiv \int_{-1}^{1} f^*(s)\, g(s)\, ds$ $\Rightarrow$ $\begin{matrix} a=-1 \\ b=1 \end{matrix}$ $w(s) = 1$

c. $\langle P_0|s^2\rangle = \int_{-1}^{1} P_0^*(s)\, s^2 ds = \int_{-1}^{1} (1)\, s^2 ds = \left.\frac{s^3}{3}\right|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \boxed{\frac{2}{3}}$

d. $\langle P_0|P_2\rangle = \int_{-1}^{1} P_0^*(s)\, P_2(s)\, ds = \int_{-1}^{1} (1)\left(\frac{3}{2}s^2 - \frac{1}{2}\right) ds = \left[\frac{s^3}{2} - \frac{s}{2}\right]_{-1}^{1} = \boxed{0}$ ← orthogonal

5. Schwartz Inequality:

a. $\boxed{|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle}$ $\Leftrightarrow$ $(\underset{\sim}{A}\cdot\underset{\sim}{B})^2 = |\underset{\sim}{A}|^2|\underset{\sim}{B}|^2\cos\theta \leq |\underset{\sim}{A}|^2|\underset{\sim}{B}|^2$

b. ⇒ Norms shrink on a nontrivial projection. (where $f \neq kg$).

E. Orthogonal Expansions

1. Two functions are orthogonal if $\langle f|g\rangle = 0$.
2. Basis functions ϕ_i are normalized if $\langle\phi_i|\phi_i\rangle = 1$.
3. Orthonormal basis: Functions that are both orthogonal & normalized.
4. Projection onto orthonormal basis: (2D example)

$$\langle\phi_1|f\rangle = \langle\phi_1|a_1\phi_1 + a_2\phi_2\rangle = a_1\langle\phi_1|\phi_1\rangle + a_2\overbrace{\langle\phi_1|\phi_2\rangle}^{0} = a_1!$$

5. For an orthonormal basis:

$$\boxed{\text{If } \langle\phi_i|\phi_j\rangle = \delta_{ij} \text{ and } f = \sum_{i=1}^{n} a_i\phi_i, \text{ then } a_i = \langle\phi_i|f\rangle}$$

6. If not normalized but orthogonal basis,

$$\boxed{f = \sum_{i=1}^{n} a_i\phi_i \quad \text{where} \quad a_i = \frac{\langle\phi_i|f\rangle}{\langle\phi_i|\phi_i\rangle}}$$

7. <u>Example</u>: a. Basis functions $\chi_n(x) = \sin(nx)$ $n = 0, 1, 2, 3, \dots$

b. $\langle f | g \rangle = \int_0^{\pi} f^*(x)\, g(x)\, dx$

c. <u>Check orthogonality</u>: $\langle \chi_n | \chi_m \rangle = \int_0^{\pi} \sin(nx)\sin(mx)\, dx = \frac{\pi}{2}\delta_{nm}$

orthogonal, but not normalized!

d. <u>Construct orthonormal basis</u>:

$$\phi_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx) \qquad n = 0, 1, 2, 3, \dots$$

e. Express $f(x) = x^2(\pi - x)$ in this basis $\phi_n(x)$.

Orthonormal projection →

$$a_n = \langle \phi_n | f(x) \rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\pi} \sin(nx)\, x^2(\pi - x)\, dx$$

f. Then

$$f(x) = x^2(\pi - x) = \sum_{n=0}^{\infty} a_n \phi_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n \sin(nx)$$

8. <u>Example</u>: Four spin-$\frac{1}{2}$ particles in triplet state.

a. $\chi_1 = \alpha\beta\alpha\alpha - \beta\alpha\alpha\alpha$, $\chi_2 = \alpha\alpha\alpha\beta - \alpha\alpha\beta\alpha$, $\chi_3 = \alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha - \alpha\beta\alpha\alpha - \beta\alpha\alpha\alpha$

b. $\langle abcd | wxyz \rangle = \delta_{aw}\delta_{bx}\delta_{cy}\delta_{dz}$

c. Create orthonormal basis:

i. $\langle \chi_1 | \chi_1 \rangle = \langle \alpha\beta\alpha\alpha | \alpha\beta\alpha\alpha \rangle - \cancelto{0}{\langle \alpha\beta\alpha\alpha | \beta\alpha\alpha\alpha \rangle} - \cancelto{0}{\langle \beta\alpha\alpha\alpha | \alpha\beta\alpha\alpha \rangle} + \langle \beta\alpha\alpha\alpha | \beta\alpha\alpha\alpha \rangle$
$= 2$

ii. Similarly $\langle \chi_2 | \chi_2 \rangle = 2$ and $\langle \chi_3 | \chi_3 \rangle = 4$

iii. Thus $\phi_1 = \frac{1}{\sqrt{2}}\chi_1$, $\phi_2 = \frac{1}{\sqrt{2}}\chi_2$ $\phi_3 = \frac{1}{2}\chi_3$ <u>orthonormal</u>

d. Expand $\chi_0 = \alpha\alpha\beta\alpha - \alpha\beta\alpha\alpha$

i. $a_1 = \langle \phi_1 | \chi_0 \rangle = -\frac{1}{\sqrt{2}}$, $a_2 = \langle \phi_2 | \chi_0 \rangle = -\frac{1}{\sqrt{2}}$, $a_3 = \langle \phi_3 | \chi_0 \rangle = \frac{1}{2} + \frac{1}{2} = 1$

$$\Rightarrow \boxed{\chi_0 = -\frac{1}{\sqrt{2}}\phi_1 - \frac{1}{\sqrt{2}}\phi_2 + \phi_3}$$

I. F. Other Properties

1. Relation between expansions and scalar products

a. $f = \sum_n a_n \phi_n \qquad g = \sum_m b_m \phi_m$

b. $\langle f | g \rangle = \sum_{n,m} a_n^* b_m \underbrace{\langle \phi_n | \phi_m \rangle}_{= \delta_{nm} \text{ if orthonormal}} = \sum_n a_n^* b_n \leftarrow$ Looks like "dot product" $\underset{\sim}{a}^* \cdot \underset{\sim}{b}$

c. For $\langle f | f \rangle = \sum_n |a_n|^2$

d. Connection to matrices: $\underset{\sim}{a} \leftrightarrow f \qquad \underset{\sim}{b} \longleftrightarrow g$

$$\langle f | g \rangle = \underset{\sim}{a}^\dagger \underset{\sim}{b} \qquad \langle f | f \rangle = \underset{\sim}{a}^\dagger \underset{\sim}{a}$$

$\nwarrow$ adjoint

2. Bessel's Inequality and Completeness

a. How do we know a set of basis functions are complete (span the space)?

b. Power series and trigonometric series are complete for expanding square integrable functions f ($\langle f | f \rangle \geq 0$, $\mathcal{L}^2$)

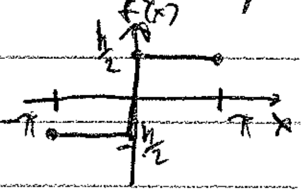
c. Test for Completeness:

Bessel's Inequality $\quad \langle f | f \rangle \geq \sum_n |a_n|^2 \quad$ where $=$ complete, $>$ incomplete

$\Rightarrow$ But, not very practical $\Rightarrow$ need to apply for all f to prove complete

3. Discontinuous Functions:

$$f(x) = \begin{cases} \frac{h}{2} & 0 < x < \pi \\ -\frac{h}{2} & -\pi < x < 0 \end{cases}$$



a. Cannot be represented by power series

b. Together, $\cos(nx)$ & $\sin(nx)$ $(n = 0, 1, 2, \ldots)$ form a complete set on $[-\pi, \pi]$, with the possibility of discontinuities.

c. $f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$

d. Since $f(x)$ is odd, all $a_n = 0$.

e. $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt = \frac{h}{n\pi} \left[1 - \cos(n\pi) \right] = \begin{cases} 0 & n \text{ even} \\ \frac{2h}{n\pi} & n \text{ odd.} \end{cases}$

Normalization factor, $w = \frac{1}{\pi}$ $\longrightarrow$

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f. Thus, $\boxed{f(x) = \frac{2h}{n\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}}$

4. Expansion of Dirac Delta Function

a. $$\delta(x-t) = \sum_{n=0}^{\infty} c_n(t)\phi_n(x) \quad \text{where } a<t<b$$

b. $$c_n(t) = \langle \phi_n(x) | \delta(x-t) \rangle = \int_a^b \phi_n^*(x)\,\delta(x-t)\,dx = \phi_n^*(t)$$

c. Thus $$\delta(x-t) = \sum_{n=0}^{\infty} \phi_n^*(t)\,\phi_n(x)$$ ← Not uniformly convergent at $x=t$.

Closure for Dirac Delta Function with respect to ϕ_n.

d. Apply to function $F(t)$: $$\int_a^b F(t)\,\delta(x-t)\,dt = \int_a^b dt \sum_m c_m \phi_m(t) \sum_{n=0}^{\infty} \phi_n^*(t)\phi_n(x)$$

where $F(t) = \sum_m c_m \phi_m(t)$

$$= \sum_{mn} c_m \phi_n(x) \underbrace{\int_a^b dt\, \phi_n^*(t)\phi_m(t)}_{=\langle \phi_n | \phi_m \rangle = \delta_{nm}} = \sum_n c_n \phi_n(x) = F(x)!$$

5. Ex: Represent $\delta(x-t)$ on basis $\phi_n(x) = \sqrt{2}\sin(n\pi x)$ on $(0,1)$ $(n=1,2,\dots)$

a. $$\delta(x-t) = \sum_{n=0}^{\infty} \phi_n^*(t)\,\phi_n(x) = \boxed{\lim_{N\to\infty} \sum_{n=1}^{N} 2\sin(n\pi t)\sin(n\pi x)}$$

G. Identity and Dirac Notation

1. $$|f\rangle = \sum_j a_j |\phi_j\rangle \underset{\uparrow}{=} \sum_j |\phi_j\rangle\langle\phi_j|f\rangle = \Big(\sum_j |\phi_j\rangle\langle\phi_j|\Big)|f\rangle$$

$a_j = \langle \phi_j | f \rangle$

2. Thus $\boxed{\mathbb{1} = \sum_j |\phi_j\rangle\langle\phi_j|}$ ket-bra sum.

⇒ resolution of the identity