Howes ①

Lecture #17 Second-Order Linear ODEs

I. Singularities

1. Second-order, linear ODEs arise frequently in physics.
2. No universally applicable closed-form solution exists, but methods can be used to form a power-series solution

A. Singular Points

1. Singularities are important for:
   a. Classifying ODEs
   b. Determine feasibility of finding a series solution.

2. Linear, Homogeneous, Second-Order ODE

$$\boxed{y'' + P(x)\, y' + Q(x)\, y = 0}$$

3. Classification of Singularities
   a. Ordinary Points: Points x_0 where $P(x)$ and $Q(x)$ are finite.

   b. Regular Singular Point: $P(x)$ or $Q(x)$ diverge as $x \to x_0$, but $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ remain finite.
   c. Irregular Singular Point (Essential): $P(x)$ or $Q(x)$ diverge as $x \to x_0$, and $(x-x_0)P(x)$ or $(x-x_0)^2 Q(x)$ diverges as $x \to x_0$.

4. Singularity at $x_0 \to \infty$:
   a. Set $x = \frac{1}{z}$, substitute into ODE, and examine limit as $z \to 0$.
   b. $w(z) = y(z^{-1})$

$$y' = \frac{d\,y(x)}{dx} = \frac{d\,y(z^{-1})}{dz}\frac{dz}{dx} = \frac{dw(z)}{dz}\left(-\frac{1}{x^2}\right) = -z^2 w' \quad z = \frac{1}{x} \quad \leftarrow \text{(w.r.t. } z)$$

   c. Similarly $\quad y'' = z^4 w'' + 2z^3 w'$

I.A.4. (Continued) Hawes ②

d. Transformation yields $\boxed{w'' + \underbrace{\frac{2z - P(z^{-1})}{z^2}}_{=P'(z)} w' + \underbrace{\frac{Q(z^{-1})}{z^4}}_{=Q'(z)} w = 0}$

e. <u>Ordinary Point at $x_0 = \infty$:</u> If $P'(z)$ & $Q'(z)$ finite as $z \to 0$

f. <u>Regular Singular Point at $x_0 = \infty$:</u> If $P'(z)$ & $Q'(z)$ diverge as $z \to 0$ but $zP'(z)$ and $z^2 Q'(z)$ remain finite

g. <u>Irregular Singular Point at $x_0 = \infty$:</u> If $P'(z)$ & $Q'(z)$ diverge as $z \to 0$ and $zP'(z)$ or $z^2 Q'(z)$ diverge as $z \to 0$.

5. <u>Ex:</u> <u>Bessel's Equation:</u> $x^2 y'' + xy' + (x^2 - n^2) y = 0$

a. $y'' + \frac{1}{x} y' + \frac{x^2 - n^2}{x^2} y = 0 \Rightarrow P(x) = \frac{1}{x}, \quad Q(x) = \frac{x^2 - n^2}{x^2}$

b. As $x_0 \to 0$, $P(x)$ & $Q(x)$ diverge, but $xP(x)$ and $x^2 Q(x)$ are finite.

$\Rightarrow$ <u>$x_0 = 0$ is a regular singular point</u>

c. for $x_0 \to \infty$, $z = \frac{1}{x}$ gives $w'' + \left(\frac{2z - z}{z^2}\right) w' + \left(\frac{1 - n^2 z^2}{z^4}\right) w = 0$

As $z \to 0$, $zP'(z)$ is finite, but $z^2 Q'(z) = \frac{1 - n^2 z^2}{z^2}$ diverges.

$\Rightarrow$ <u>$x_0 = \infty$ is an irregular singular point</u>

6. <u>NOTE:</u> Determination of nature of singularities is important to determine if a series solution is possible (Fuchs Thm).

II. <u>Frobenius' Method for Series Solutions</u>

A. <u>General Procedure:</u>

1. Linear, 2nd-Order, Homogeneous ODE: $\boxed{\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0}$

2. Expand y as power series about point $x = 0$.

3. <u>Ex:</u> Linear Simple Harmonic Oscillator: $\boxed{y'' + \omega^2 y = 0}$

a. <u>Power Series:</u> $\boxed{y(x) = x^s (a_0 + a_1 x + a_2 x^2 + \dots)}$ <u>NOTE:</u> s need not be an integer!

II.A.3. (Continued) $\swarrow a_0 \neq 0.$ Hawes ③

b. $y(x) = \sum_{j=0}^{\infty} a_j x^{s+j} \Rightarrow y''(x) = \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2}$

c. Thus $\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + \omega^2 \sum_{i=0}^{\infty} a_i x^{s+i} = 0$

d. To collect powers of x, $j = i+2$ has same power of x, so

$\sum_{j=0}^{\infty} f_j = \sum_{i=-2}^{\infty} f_i = f_{-2} + f_{-1} + \sum_{i=0}^{\infty} f_i$, thus

$$\underbrace{a_0 s(s-1)}_{=0} x^{s-2} + \underbrace{a_1 (s+1) s}_{=0} x^{s-1} + \sum_{i=0}^{\infty} \underbrace{\left[a_{i+2} (s+i+2)(s+i+1) + \omega^2 a_i\right]}_{=0} x^{s+i} = 0$$

e. By uniqueness of power series solution, each coefficient must vanish!

x^{s-2} f. Indicial Equation: $a_0 s(s-1) = 0$ where $a_0 \neq 0$ by definition

Solutions: $\boxed{s=0,\ s=1}$ $\Rightarrow$ Series solution starts with x^0 or x^1 term!

x^{s-1} g. $a_1 (s+1) s = 0$ i. If $s=0$, a_1 is unconstrained

ii. If $s=1$, $a_1 = 0$.

x^{s+i} terms h. $a_{i+2}(s+i+2)(s+i+1) + \omega^2 a_i = 0$

i. $$\boxed{a_{i+2} = \frac{-\omega^2}{(s+i+2)(s+i+1)} a_i}$$ Recurrence Relation

ii. Gives $a_2, a_4, a_6 \ldots$ in terms of a_0 $\quad a_0 \neq 0$

and $a_3, a_5, a_7, \ldots$ in terms of a_1 $\quad$ $\boxed{\text{Let us take } a_1 = 0!}$

i. $\boxed{s=0 \text{ solution}}$ $a_{i+2} = \frac{-\omega^2}{(i+2)(i+1)} a_i$

$\Rightarrow a_2 = -\frac{\omega^2}{2!} a_0 \qquad a_4 = \frac{-\omega^2}{3 \cdot 4} a_0 = \frac{-\omega^2}{3 \cdot 4}\left(\frac{-\omega^2}{2!} a_0\right) = \frac{+\omega^4}{4!} a_0$

$\Rightarrow$ In general, $\boxed{a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0}$

Solution: $y(x) = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \ldots\right] = \boxed{a_0 \cos(\omega x) = y(x)}$

j. $\boxed{S=1 \text{ Solution}}$ $a_{i+2} = \dfrac{-\omega^2}{(i+3)(i+2)} a_i$

$$\Rightarrow a_2 = -\frac{\omega^2}{3!} a_0 \Rightarrow a_4 = \frac{+\omega^4}{5!} a_0 \Rightarrow a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

<u>Solution</u>: $y(x) = a_0 x\left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \dots\right] = \frac{a_0}{\omega}\left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \dots\right]$

$$\boxed{y(x) = \frac{a_0}{\omega} \sin(\omega x)}$$

k. Thus, general solution: $\boxed{y(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x)}$

4. <u>Cautionary Note about Series Solutions using Frobenius' Method</u>:
   a. <u>Always</u> substitute solutions back into original ODE as a check!
   b. Need to check that series converges over region of interest.

B. <u>Additional Considerations</u>
1. Expanding about a point $x_0 \neq 0$, $y(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^{s+j}$, $a_0 \neq 0$.
   a. Legendre, Chebyshev, and Hypergeometric eqs. use $x_0 = 1$.
   b. x_0 should <u>not</u> be an irregular singularity!

C. <u>Ex: Bessel's Equation</u> $x^2 y'' + x y' + (x^2 - n^2) y = 0$

1. Let $y(x) = \sum_{j=0}^{\infty} a_j x^{s+j}$ $a_0 \neq 0$

2. $\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=0}^{\infty} a_j (s+j) x^{s+j} + \sum_{i=0}^{\infty} a_i x^{s+i+2} - \sum_{j=0}^{\infty} a_j n^2 x^{s+j} = 0$

x^s 3. <u>Indicial Equation</u> $j=0$ term: $a_0 \neq 0$!

$$a_0 s(s-1) + a_0(s) - a_0 n^2 = a_0[s^2 - \cancel{s} + \cancel{s} - n^2] = \boxed{a_0(s^2 - n^2) = 0}$$

a. Solutions $\boxed{s = +n, \ s = -n}$

x^{s+1} 4. a. $a_1(s+1)s + a_1(s+1) - a_1 n^2 = \boxed{a_1(s+1-n)(s+1+n) = 0}$

II C4. (Continued)

b. For either $S=+n$ or $S=-n$, solution requires $\boxed{a_1=0}$.

x^{S+i+2} 5. Recurrence Relation: $(j=i+2)$:

a. $a_j\left[(S+j)(S+j-1)+S+j-n^2\right]+a_{j-2}=0$

6. $\boxed{\text{For } S=n}$, we obtain: $\boxed{a_{j+2}=\dfrac{-1}{(j+2)(2n+j+2)}a_j}$

7. We can obtain the general form

$$a_{2p}=(-1)^p\frac{a_0\,n!}{2^{2p}\,p!(n+p)!}\quad, \text{yielding}\quad Y(x)=a_0 2^n n!\sum_{j=0}^{\infty}(-1)^j\frac{1}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}$$

$\boxed{Y(x)=J_n(x)}$ for $a_0=\dfrac{1}{2^n n!}$

8. For $S=-n$, $a_{j+2}=\dfrac{-1}{(j+2)(-2n+j+2)}a_j$

a. But, when $-2n+j+2=0$, or $j=2(n-1)$, coefficient diverges!

b. For integral n, Frobenius' method fails to find a second series solution!

9. Thus $\boxed{\text{Frobenius' method for series solution does not always yield two linearly independent solutions.}}$

D. Regular vs. Irregular Singularities

1. Consider two equations: (A) $Y''-\dfrac{6}{x^2}Y=0$

(B) $Y''-\dfrac{6}{x^3}Y=0$

Substitute: $Y=\sum_{j=0}^{\infty}a_j x^{S+j}$

2. For (A), indicial equation is $S(S-1)-6=0 \Rightarrow (S-3)(S+2)=0$

a. Solutions $S=3$, $S=-2$

b. All terms have same power in x (ODE is homogeneous in x & Y), so there is no recurrence relation. $\Rightarrow$ Solutions: $Y_1=C_1x^3$, $Y_2=C_2x^{-2}$

II. D. (Continued) Hawes 6

3. For (B), $\sum_{j=0} a_j (s+j)(s+j-1) x^{s+j-2} - 6 \sum_{j=0}^{\infty} a_j x^{s+j-3} = 0$

a. Indicial equation: Lowest order x^{s-3}: $-6a_0 = 0$, but $a_0 \neq 0$!
$\Rightarrow$ No solution!

b. Expansion about irregular singular point at $x=0$ fails!

E. Fuchs Theorem: We can always obtain at least one power-series solution if we expand about an ordinary point or a regular singular point.

III. Obtaining a Second, Linearly Independent Solution

A. Determining Linear Independence of Solutions: Wronskian

1. Consider a set of n function ϕ_i

a. The set is linearly dependent if one can solve
$$\sum_{i=1}^{n} k_i \phi_i(x) = 0 \quad (\text{with not all } k_i = 0.)$$

b. If only solution is $k_i = 0$, functions are linearly independent.

2. Mutually orthogonal functions are linearly independent.

a. $S = \langle \sum_i k_i \phi_i | \sum_j k_j \phi_j \rangle = \sum_{ij} k_i^* k_j \underbrace{\langle \phi_i | \phi_j \rangle}_{=\delta_{mn}} = \sum_i |k_i|^2 > 0$ unless $k_i = 0$!
$\Rightarrow$ linearly independent!

3. Wronskian:

a. Assuming differentiable functions ϕ_i, $\sum_i k_i \phi'(x) = 0$
$\sum_i k_i \phi''(x) = 0$, etc.

b. Do this until you generate n equations.

III A. 3. (Continued)

c. This set of n homogeneous equations has a nontrivial solution (not $K_i = 0$) only if determinant vanishes!

d. Wronskian:
$$\boxed{W \equiv \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & & \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} = 0}$$

e. 1) If Wronskian $W \neq 0$, functions ϕ_i are linearly independent.

2) Only if $W = 0$ over entire range of x are ϕ_i linearly dependent.

4. Ex: Linear Oscillator: $\phi_1 = \sin \omega x$, $\phi_2 = \cos \omega x$

a. $$W = \begin{vmatrix} \sin \omega x & \cos \omega x \\ \omega \cos \omega x & -\omega \sin \omega x \end{vmatrix} = -\omega(\cancel{\sin^2 \omega x + \cos^2 \omega x}^{\,1}) = -\omega \neq 0 \Rightarrow$$ linearly independent.

B. Number of Solutions

1. Theorem: A linear, 2nd-order homogeneous ODE has at most two linearly independent solutions. An n-th order, linear, homogeneous ODE has at most n linearly independent solutions.

Proof in text.

C. Constructing a Second, Linearly Independent Solution

1. For $\boxed{y'' + P(x)\,y' + Q(x)\,y = 0}$ with solution $y = y_1(x)$.

2. A second solution can be constructed by
$$\boxed{y_2(x) = y_1(x) \int^x \left[\frac{\exp\left(-\int^{x_2} P(x_1)\,dx_1\right)}{[y_1(x_2)]^2} \right] dx_2}$$

3. A common special case has $P(x) = 0$, so
$$\boxed{y_2(x) = y_1(x) \int^x \frac{dx_2}{[y_1(x_2)]^2}}$$

4. NOTE: Any linear, 2nd-order homogeneous ODE can be transformed into a form with $P(x) = 0$.

5. Ex: Simple Harmonic Oscillator. $y'' + y = 0$

a. One solution: $\boxed{y_1(x) = \sin x}$

b. $y_2(x) = \sin x \int^x \frac{dx_2}{\sin^2 x_2} = \sin x\left[-\cot x + \cancel{C}\right] = \sin x \frac{-\cos x}{\sin x} = -\cos x$

$C \to 0 \to$ only adds $C y_1$

c. Thus $\boxed{y_2(x) = \cos x}$

D. Series Form of Second Solution

1. We can expand $P(x) = \sum_{i=-1}^{\infty} p_i x^i$ and $Q(x) = \sum_{j=-2}^{\infty} q_j x^j$

to compute second solution by integrating the form for $y_2(x)$ [above in III.C.2] term by term.

2. Series form of Second solution

$$\boxed{y_2(x) = y_1(x) \ln|x| + \sum_{j=-n}^{\infty} d_j x^{j+\alpha}}$$

where n is an integer.