Lecture #18: Homogeneous & Inhomogeneous ODEs, Sturm-Liouville Theory

I. Second Solution for Linear Homogeneous Second-Order ODE (Continued)

A. Example: Bessel's Equation for $n=0$.

1. $$x^2 y'' + x y' + x^2 y = 0$$

2. Putting it into standard form, $y'' + \underbrace{\frac{1}{x}}_{=P(x)} y' + \underset{\uparrow\, Q(x)=1}{} y = 0$

3. From the Frobenius' Method Solution, indicial equation is $a_0(s^2 - \cancel{n^2}^{\,0}) = 0$ (from Lec #17, II C. 3.)
   a. So, $s^2 = 0 \Rightarrow \boxed{s=0}$
   b. Resulting series solution is $y_1(x) = J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - O(x^6)$

4. Now, use integral form for 2nd solution (Lec #17, III, C2.),
   a. $$\boxed{y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^{x_2} P(x_1)\, dx_1\right]}{[y_1(x_2)]^2}\, dx_2}$$ Wronskian Double Integral

   b. $$= J_0(x) \int^x \frac{\exp\left[-\int^{x_2} \left(\frac{dx_1}{x_1}\right)\right]}{\left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64} - \dots\right]^2}\, dx_2$$

   c. Numerator: $\exp\left[-\int^{x_2} \frac{dx_1}{x_1}\right] = \exp\left[-\ln x_2 + C_0\right] = \frac{e^{C_0}}{x_2} = \frac{C_1}{x_2}$

   d. Denominator: Use binomial expansion $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \dots$
   $$\left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64} \dots\right] = 1 - 2\left(\frac{-x_2^2}{4} + \frac{x_2^4}{64} \dots\right) + 3\left(\frac{-x_2^2}{4} + \frac{x_2^4}{64} \dots\right)^2 - \dots$$
   $\delta = -\frac{x_2^2}{4} + \frac{x_2^4}{64} - \dots$
   $$= 1 + \frac{x_2^2}{2} + \frac{5}{32} x_2^4 + \dots$$

   e. Thus
   $$y_2(x) = J_0(x) \int^x \frac{C_1}{x_2}\left[1 + \frac{x_2^2}{2} + \frac{5}{32} x_2^4 + \dots\right] dx_2 = \boxed{C_1 J_0(x) \left\{\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + \dots\right\}}$$

5. Standard form of Second Solution of Bessel's Equation: <u>Neumann Function</u>
$$Y_0(x) = \frac{2}{\pi}\left[\ln x - \ln 2 + \gamma\right] J_0(x) + \frac{2}{\pi}\left\{\frac{x^2}{4} - \frac{3x^4}{128} + \dots\right\}$$

I. A (Continued)

6. We can construct the Neumann function from a linear combination of our solutions $y_1(x)$ & $y_2(x)$
   a. $C_1 = \frac{2}{\pi}$, add $\frac{2}{\pi}[-\ln 2 + \gamma] J_0(x)$
   b. Multiply series expansion of $J_0(x)$ in Right-hand term.

7. Note: a. $Y_0(x)$ diverges at the origin $\Rightarrow$ irregular solution
   b. $J_0(x)$ converges at the origin $\Rightarrow$ regular solution

B. Summary of Solutions to Linear, Homogeneous, Second-Order ODEs

1. Two solutions $y_1(x)$ and $y_2(x)$ form complete solution

$$\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x)}$$

2. Point of expansion must be ordinary or regular singular point
3. By Fuch's Theorem, one solution can always be obtained by Frobenius!
4. Wronskian double integral yields 2nd, linearly independent solution.
5. No third linearly independent solution exists.

II. Inhomogeneous Linear ODEs

1. General Form: $\boxed{y'' + P(x) y' + Q(x) y = F(x)}$

2. We assume homogeneous eq, $[F(x) = 0]$ has been solved for $y_1(x)$ & $y_2(x)$

A. Variation of Parameters Method

1. Write the particular solution as $\boxed{y_p(x) = \underset{\uparrow}{u_1(x)}\, y_1(x) + \underset{\uparrow}{u_2(x)}\, y_2(x)}$
   Functions of x

2. $y_p' = u_1 y_1' + u_2 y_2' + (y_1 u_1' + y_2 u_2')$
3. We choose $u_1(x)$ & $u_2(x)$ such that $\boxed{y_1 u_1' + y_2 u_2' = 0}$
4. Thus

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

II A. (Continued)

5. Substituting into ODE

a. $\left[U_1 y_1'' + U_2 y_2'' + U_1' y_1' + U_2' y_2'\right] + P(x)\left[U_1 y_1' + U_2 y_2'\right] + Q(x)\left[U_1 y_1 + U_2 y_2\right] = F(x)$

b. $U_1 \underbrace{\left[y_1'' + P(x) y_1' + Q(x) y_1\right]}_{=0 \text{ (Homogeneous Solution)}} + U_2 \underbrace{\left[y_2'' + P(x) y_2' + Q(x) y_2\right]}_{=0} + U_1' y_1' + U_2' y_2' = F(x)$

6. a. Thus, we are left with two equations to solve for U_1' & U_2'

$$y_1 U_1' + y_2 U_2' = 0$$
$$y_1' U_1' + y_2' U_2' = F(x)$$

b. <u>NOTE</u>:
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \quad \text{since } y_1 \text{ \& } y_2 \text{ are linearly independent solutions}$$

c. Thus, there is a <u>unique, nontrivial solution</u> for U_1' & U_2'.

d. Simply integrate to obtain <u>$U_1(x)$ & $U_2(x)$</u> and thus <u>particular solution</u> $y_p(x)$.

6. <u>Ex</u>: <u>Inhomogeneous ODE</u> $(1-x)y'' + xy' - y = (1-x)^2$

a. Solutions to homogeneous ODE are $y_1 = x$, $y_2 = e^x$.

b. Thus
$$x U_1' + e^x U_2' = 0$$
$$U_1' + e^x U_2' = F(x) = (1-x) \leftarrow \text{Standard form!}$$

c. Solution yields
$$U_1' = 1 \Rightarrow U_1 = x$$
$$U_2' = -xe^{-x} \Rightarrow U_2 = (x+1)e^{-x}$$

d. Form particular Solution $y_p(x) = x[x] + (x+1)e^{-x}[e^x] = x^2 + x + 1$ ← (can eliminate x because $y_1 = x$.)

e. General Solution is homogeneous solutions plus particular solution

$$\boxed{y(x) = C_1 x + C_2 e^x + x^2 + 1}$$

III <u>Nonlinear Differential Equations</u> (NDEs)

1. Some physics involves <u>nonlinear differential equations</u>

a. <u>Ex</u>: Navier-Stokes equations for hydrodynamics.

III. (Continued)

2. Nonlinearity is responsible for development of turbulence
3. NDEs also lead to chaos, where a system is so sensitive to initial conditions as to be unpredictable
4. Analytically, some NDEs can be solved by clever tricks that convert them to linear ODE's (Ex: Bernoulli & Riccati Eq's in text).
5. In general, behavior of NDEs is studied numerically.

IV. Sturm-Liouville Theory

A. Introduction

1. Many physics problems with ODEs have two properties:
   a. Boundary Conditions must be satisfied. ⇒ Define Hilbert Space
   b. Parameter must be chosen to satisfy BCs. ⇒ Eigenvalue.

2. Ex: Standing Waves on a String Clamped at Ends
   a. $\frac{d^2\psi}{dx^2} + k^2\psi = 0$ where $\psi(x)$ is amplitude and k is parameter.

   b. Hilbert Space: Differentiable functions with $\psi = 0$ at ends.
   c. Eigenvalue Equation: $\mathcal{L}\psi = k^2\psi$ where $\mathcal{L} = -\frac{d^2}{dx^2}$

   d. We can solve this with a general vector space approach
      i. Define a basis for Hilbert Space
      ii. Define Scalar product
      iii. Expand $\mathcal{L}$ and ψ in terms of basis
      iv. Solve resulting matrix equation
   e. But, since we can solve the ODE, we can do this more efficiently!

3. Ex: Standing Waves: ODE Solution with BCs.
   a. $\frac{d^2\psi}{dx^2} + k^2\psi = 0$ with $\psi(0) = \psi(l) = 0$.

b. General Solution: $\boxed{\psi(x) = A\sin kx + B\cos kx}$

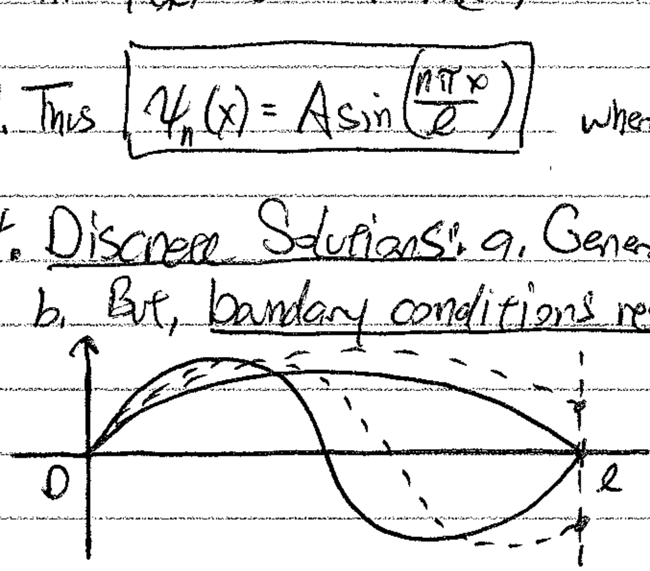
c. Apply BCs: i. $\psi(0) = 0 = A(0) + B(1) \Rightarrow \boxed{B=0}$

ii. $\psi(\ell) = 0 = A\sin(k\ell) \Rightarrow k\ell = n\pi$ for any integer n.

d. Thus $\boxed{\psi_n(x) = A\sin\left(\frac{n\pi x}{\ell}\right)}$ where eigenvalue $\boxed{k^2 = \frac{n^2\pi^2}{\ell^2}}$

4. Discrete Solutions: a. General Solution valid for any A, B, k.

b. But, boundary conditions restrict solutions to discrete k values.



c. Amplitude of Solution remains arbitrary $\Rightarrow$ homogeneous ODE.

5. Normalizing Solutions:

a. Define Scalar Product $\langle f|g\rangle = \int_0^\ell f^*(x)g(x)\,dx$

b. Require $\langle \psi_n|\psi_n\rangle = 1$.

c. Thus, we obtain $\boxed{\psi_n(x) = \sqrt{\frac{2}{\ell}}\,\sin\left(\frac{n\pi x}{\ell}\right)}$

6. Properties of Solution: Hermitian operator $\mathcal{L}$

a. Hermitian property depends on $\mathcal{L}$ and definition of $\langle f|g\rangle$.

b. Since this operator is Hermitian, it implies real eigenvalues and a complete basis of orthogonal eigenfunctions

B. Hermitian Operators

1. Sturm-Liouville Theory $\mathcal{L}\psi(x) = \lambda\psi(x)$

where $\mathcal{L}(x) = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$

a. We want to identify conditions under which $\mathcal{L}$ is Hermitian

2. Self-Adjoint ODEs

a. $\mathcal{L}$ is self-adjoint if $\boxed{p_0'(x) = p_1(x)}$

b. Thus $\mathcal{L}(x) = p_0 \frac{d^2}{dx^2} + p_0' \frac{d}{dx} + p_2 = \frac{d}{dx}\left[p_0 \frac{d}{dx}\right] + p_2$

c. Define $\langle f | g \rangle = \int_a^b f^*(x)\, g(x)\, dx$, so

$$\langle f | \mathcal{L} g \rangle = \int_a^b \left[f^*(p_0 g')' + f^* p_2 g\right] dx = \left[f^* p_0 g'\right]_a^b + \int_a^b \left(-f^{*\prime} p_0 g' + f^* p_2 g\right) dx$$

$u = f^*$ $\quad dv = (p_0 g')'$ $\qquad\qquad$ $u = p_0 f^{*\prime}$ $\quad dv = g'$

$du = f^{*\prime}$ $\quad v = p_0 g'$ $\qquad\qquad$ $du = (p_0 f^{*\prime})'$ $\quad v = g$

$$= \left[f^* p_0 g' - f^{*\prime} p_0 g\right]_a^b + \int_a^b \underbrace{\left[(p_0 f^{*\prime})' + p_2 f^*\right]}_{= \mathcal{L} f^*} g\, dx = \left[\ldots\right]_a^b + \underbrace{\langle \mathcal{L} f | g \rangle}_{= \langle f | \mathcal{L}^\dagger g \rangle}$$

d. If boundary terms vanish, then $\mathcal{L}$ is self-adjoint, $\mathcal{L} = \mathcal{L}^\dagger$.

e. Classes of Boundary Conditions: i. Dirichlet BCs: $f(a) = g(a) = 0$, $f(b) = g(b) = 0$.

ii. Neumann BC's: $f'(a) = g'(a) = 0$, $f'(b) = g'(b) = 0$.

iii. Periodic B.C.'s: $f(a) = f(b)$, $g(a) = g(b)$.

f. Orthogonality: i. If f and g are eigenfunctions, with λ_f and λ_g, then

ii. $\langle f | \mathcal{L} g \rangle - \langle \mathcal{L} f | g \rangle = (\lambda_g - \lambda_f)\langle f | g \rangle = \left[f^* p_0 g' - f^{*\prime} p_0 g\right]_a^b$

iii. Then, if BC's vanish, $\langle f | g \rangle = 0$ for $\lambda_g \neq \lambda_f \Rightarrow$ Orthogonality!

C. Making an ODE Self-Adjoint

1. If an operator is not self-adjoint $[p_0'(x) = p_1(x)]$, it can be multiplied by a factor to make it self-adjoint.

a. $W(x)\,\mathcal{L}(x)\,\psi(x) = W(x)\,\lambda\,\psi(x)$ $\quad$ where $\boxed{W(x) = p_0^{-1} \exp\left[\int \frac{p_1(x)}{p_0(x)} dx\right]}$

Same eigenfunctions and eigenvalues.

IV. C. (Continued)

2. a. Then $w(x)\mathcal{L}(x) = \bar{p}_0 \frac{d^2}{dx^2} + \bar{p}_1 \frac{d}{dx} + w(x) p_2(x)$

where $\bar{p}_0 = \exp\left(\int \frac{p_0}{p_1} dx\right)$ and $\bar{p}_1 = \frac{p_0}{p_1} \exp\left(\int \frac{p_0}{p_1} dx\right)$

b. Since $\bar{p}_0' = \bar{p}_1$, then <u>$w(x)\mathcal{L}(x)$ is self-adjoint</u>!

3. <u>Scalar Product</u>: $\langle f | w\mathcal{L} g \rangle = \int_a^b f^*(x) w(x) \mathcal{L} g(x)\, dx$

a. We can show this becomes $= \left[f^* \bar{p}_0 g' - f^{*\prime} \bar{p}_0 g \right]_a^b + \underbrace{\int_a^b w(x) \mathcal{L} f^*(x)\, g(x)\, dx}_{= \langle w\mathcal{L} f | g \rangle}$

b. But, this is equivalent to

$\langle f | \mathcal{L} g \rangle = \langle \mathcal{L} f | g \rangle$ for $\langle f | g \rangle \equiv \int_a^b f^*(x) g(x)\, \underset{\uparrow}{w(x)}\, dx$

weight function.

c. Can show eigenfunctions are <u>orthogonal</u>!

4. <u>Summary</u>:

a. Self-adjoint $\mathcal{L}$ ($p_0' = p_1$), then it is Hermitian if

i. Scalar product has uniform weight

ii. Boundary conditions remove end points.

b. Non-self adjoint $\mathcal{L}$ ($p_0' \neq p_1$), can be made Hermitian if

i. Scalar product includes appropriate weight factor $w(x)$

ii. Boundary conditions remove end points

5. <u>Ex</u>: <u>Laguerre Functions</u>: $\mathcal{L} = x \frac{d^2}{dx^2} + (1-x) \frac{d}{dx}$ for $0 \leq x < \infty$.
and $\psi(\infty) = 0$.

a. $p_0' = 1 \neq p_1$, so not self-adjoint.

b. $w(x) = p_0^{-1} \exp\left[\int \frac{p_1}{p_0} dx\right] = \frac{1}{x} \exp\left[\int \frac{1-x}{x} dx\right] = \frac{1}{x} \exp\left[\ln x - x\right] = \frac{1}{x} x e^{-x}$

$\boxed{w(x) = e^{-x}}$

c. Thus $\langle f | g \rangle = \int_0^\infty f^*(x) g(x) e^{-x} dx$

d. <u>B.C.s</u>: $\left[x e^{-x} (f^* g' - f^{*\prime} g) \right]_0^\infty = 0$

$\Big\}$ <u>Hermitian!</u>

IV. D. ODE Eigenvalue Problem: Legendre Equation Hawes (8)

1. $\mathcal{L}\, y(x) = -(1-x^2)y'' + 2x\,y' = \lambda y \qquad -1 \le x \le 1$

2a. Note: ODE has regular singular points at $x = \pm 1$.

b. Requirement that y is nonsingular at $x = \pm 1 \Rightarrow$ Boundary Conditions!

3. Frobenius' Method: $y = \sum_{j=0}^{\infty} a_j x^{s+j}$

a. Indicial equation $s(s-1) = 0 \Rightarrow \boxed{s=0}, \boxed{s=1}$

b. $\boxed{s=0}$ solution i. $a_1 = 0$ can be chosen

ii. Recurrence relation $a_{j+2} = \dfrac{j(j+1) - \lambda}{(j+1)(j+2)} a_j$ ← eigenvalue

c. $\boxed{s=1}$ solution i. $a_1 = 0$

ii. Recurrence relation $a_{j+2} = \dfrac{(j+1)(j+2) - \lambda}{(j+2)(j+3)} a_j$

4. Examine Convergence:

a. $s=0$ solution: Ratio test $\lim_{j\to\infty} \dfrac{u_{j+2}}{u_j} = \lim_{n\to\infty} \dfrac{j(j+1)-\lambda}{(j+1)(j+2)} \dfrac{x^{j+2}}{x^j} = x^2 \le 1$ on $-1 \le x \le 1$.

b. Thus Converges for $|x| < 1$, but is indeterminate for $|x| = 1$.

c. The more sensitive Gauss test proves divergence for $|x| = 1$.

5. Eliminate divergence by truncating infinite series:

a. Choosing $\lambda = l(l+1)$ truncates series at $a_{l+2} = 0$!

b. Thus, solutions with $\boxed{\lambda = l(l+1)}$ converge over $-1 \le x \le 1$.

c. Solutions with $\lambda = l(l+1)$ are polynomials of degree l.

6. $s=0$ solutions y_l where l is even, $s=1$ solutions y_l with l odd

7a. The ODE operator $\mathcal{L}$ is self-adjoint.

b. Coefficient $p_0 = -(1-x^2)$ vanishes at $|x| = 1$, so boundary terms $\left[f^* p_0 g' - f^{*\prime} p_0 g\right]_{-1}^{1} = 0 \Rightarrow \boxed{\mathcal{L} \text{ is Hermitian}}$