Lecture #2: Series Expansion of Functions, Vectors, Complex Functions

I. Operations of Series Expansions of Functions

A. Basic Idea:

1. A number of tricks can be used to more easily obtain a series representation of a function, or to improve the convergence of a series (either rate or range of convergence)
2. Some tricks (differentiation or integration) depend on the fact that the expansion depends on a variable.

B. Example 1: Integration

1. To obtain expansion for $f(x) = \ln(1+x)$, we may integrate term by term the expansion for $\frac{1}{1+x}$.

a. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ (by Binomial Expansion)

b. $\ln(1+x) = \int_0^x \frac{dy}{1+y} = \int_0^x (1 - y + y^2 - y^3 + \ldots)\,dy = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

C. Example 2: Obtain expansion $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots$

1. Direct application of Maclaurin expansion is not helpful

$$f(x) = \sin^{-1} x \qquad f'(x) = \frac{1}{\sqrt{1-x^2}} \quad \ldots$$

2. Instead, take $\sin y = x$ (so $y = \sin^{-1} x$)
3. $d(\sin y) = \cos y\, dy = dx \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$
4. $\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$
5. $y = \int \frac{dy}{dx}\,dx \Rightarrow y = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \left(1 + \frac{t^2}{2} + \frac{3t^4}{8} + \ldots\right)dt = \boxed{x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots}$
6. Binomial Thm $(1-t^2)^{-1/2} = 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \ldots$

I. (Continued)

D. Example 3: Improving Convergence Rate

1. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

$\boxed{\text{Converges as } \frac{1}{n}}$ (slow)

2. Multiply by $(1+a_1 x)\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} + a_1 \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n}}_{= \sum_{m=2}^{\infty} \frac{(-1)^{m-2} x^m}{(m-1)}}$

3. $= x + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{x^n}{n} - \frac{a_1 x^n}{(n-1)} \right)$

$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{[n-1-a_1 n]}{n(n-1)} x^n$ → Choose a_1 to cancel n in numerator $\Rightarrow a_1 = 1$

4. Rearranging solution, we obtain

$\ln(1+x) = \left(\frac{x}{1+x}\right)\left(1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} x^n\right)$ $\boxed{\text{Converges as } \frac{1}{n^2}}$ Faster!

E. Partial Fraction Expansion

1. For $g(x)$ and $h(x)$ two polynomials in x, (with $h(x)$ a higher degree), if $h(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$, we can write

$$\frac{g(x)}{h(x)} = \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \ldots + \frac{c_n}{x-a_n}$$

2. You may also leave a quadratic factor in $h(x)$ (to avoid imaginary a_n).

Thus, term has form. $\frac{ax+b}{x^2+px+q}$

Lecture #2 (Continued)

I. E. (Continued)

3. Example: Let $f(x) = \frac{k^2}{x(x^2+k^2)} = \frac{c}{x} + \frac{ax+b}{x^2+k^2}$

a. $\frac{k^2}{x(x^2+k^2)} = \frac{c(x^2+k^2)+x(ax+b)}{x(x^2+k^2)} = \frac{(a+c)x^2+bx+ck^2}{x(x^2+k^2)}$

b. Equate coefficients of powers in numerator (uniqueness of power series)

$a+c=0 \quad \boxed{b=0} \quad \boxed{c=1} \quad \Rightarrow \boxed{a=-1}$

c. Thus $\boxed{f(x) = \frac{1}{x} - \frac{x}{x^2+k^2}}$

F. Euler Transformation: Improving Range of Convergence

1. $f(x) = \sum_{n=0}^{\infty} (-1)^n c_n x^n = \frac{1}{1+x} \sum_{n=0}^{\infty} (-1)^n a_n \left(\frac{x}{1+x}\right)^n$ where $a_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} c_{n-j}$

2. Can expand limited range of convergence to wider range ($|x|<1$ to $x<\infty$)

G. Important Series:

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad -\infty < x < \infty$

2. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad -\infty < x < \infty$

3. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \qquad -\infty < x < \infty$

4. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad -1 \le x < 1$

5. $(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \qquad -1 < x < 1$

II. Vectors

A. Define: 1. Scalar: Quantity with magnitude only

2. Vector: Quantity with magnitude and direction

3. Vector Fields: Vectors defined over a region (Magnetic Field, etc.

B. Basic Properties:

1. Addition is commutative $(\underset{\sim}{A}+\underset{\sim}{B}=\underset{\sim}{B}+\underset{\sim}{A})$

and associative $(\underset{\sim}{A}+\underset{\sim}{B})+\underset{\sim}{C}=\underset{\sim}{A}+(\underset{\sim}{B}+\underset{\sim}{C})$

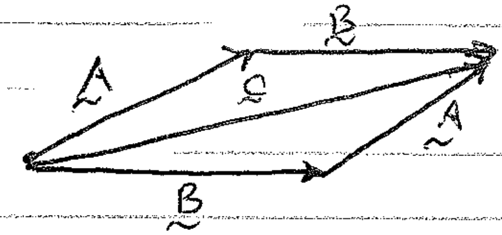
2. Subtraction: $\underset{\sim}{A}-\underset{\sim}{B}=\underset{\sim}{A}+(-1)\underset{\sim}{B}$

3. Multiplication by a Scalar:

a. $k\underset{\sim}{A}$ is same direction, but magnitude times k

b. $(-1)\underset{\sim}{A}$ is same magnitude, but opposite direction

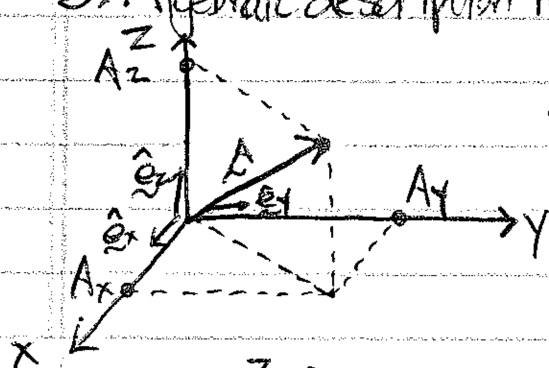
4. Vectors are independent of the coordinate system (Geometric)



$\underset{\sim}{A}+\underset{\sim}{B}=\underset{\sim}{C}$

$\underset{\sim}{B}+\underset{\sim}{A}=\underset{\sim}{C}$

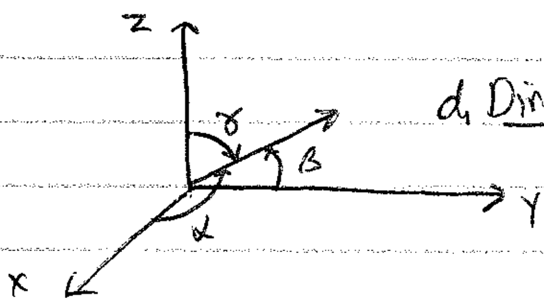
5. Algebraic description requires specifying a coordinate system.



a. Components: A_x, A_y, A_z

b. $\underset{\sim}{A}=A_x\,\underset{\sim}{\hat{e}}_x+A_y\,\underset{\sim}{\hat{e}}_y+A_z\,\underset{\sim}{\hat{e}}_z$

c. Magnitude: $|\underset{\sim}{A}|=A=\sqrt{A_x^2+A_y^2+A_z^2}$



d. Direction Cosines: $A_x=A\cos\alpha$

$A_y=A\cos\beta$

$A_z=A\cos\gamma$

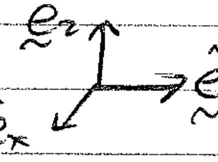
$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Lecture #2 (Continued)

II. B. (Continued)

6. Displacement vector (from origin) $\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z$

7. Unit vectors:



a. Unit magnitude

b. Direction along coordinate axes

c. Unit vector span the space, forming a complete basis.

d. 2D space [all real values of (x,y)] is $\mathbb{R}^2$

3D space [all real values of (x,y,z)] is $\mathbb{R}^3$

C. Dot Product

1. Projection: $\mathbf{A} = \underbrace{A_x\,\hat{\mathbf{e}}_x}_{} + A_y\,\hat{\mathbf{e}}_y + A_z\,\hat{\mathbf{e}}_z$

Projection of $\mathbf{A}$ along $\hat{\mathbf{e}}_x$ direction.

2. Dot Product: a. $\mathbf{A}\cdot\mathbf{B} = (A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z)\cdot(B_x\hat{\mathbf{e}}_x + B_y\hat{\mathbf{e}}_y + B_z\hat{\mathbf{e}}_z)$

$= A_x B_x + A_y B_y + A_z B_z$

since $\hat{\mathbf{e}}_x\cdot\hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y\cdot\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z\hat{\mathbf{e}}_z = 1$

and $\hat{\mathbf{e}}_x\cdot\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_y\cdot\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z\cdot\hat{\mathbf{e}}_x = 0.$

b. $\mathbf{A}\cdot\mathbf{B} = \sum_i A_i B_i$

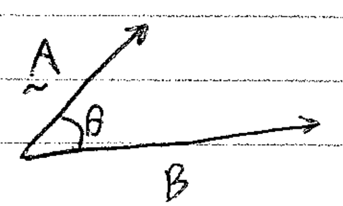
c. Commutative: $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

Associative: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$

and Distributive: $\mathbf{A}\cdot(k\,\mathbf{B}) = k(\mathbf{A}\cdot\mathbf{B})$

d. $\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2$

3.



a. $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$

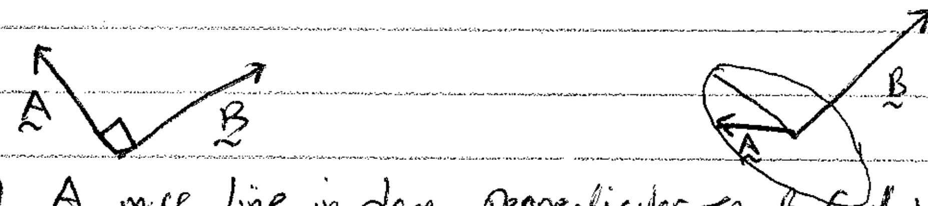
b. $|\mathbf{A}\cdot\mathbf{B}| \le |\mathbf{A}|\,|\mathbf{B}|$

II. (Continued)

D. Orthogonality

1. $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are orthogonal if and only if $\underset{\sim}{A} \cdot \underset{\sim}{B} = 0$

2. In 2D, this means vectors are perpendicular



3. In 3D, $\underset{\sim}{A}$ must line in plane perpendicular to $\underset{\sim}{B}$ (and vice versa).

III. Complex Numbers and Functions

Although physical quantities must be real, certain physical effects (such as Landau damping in plasma physics) require the use of complex analysis to be described mathematically. In some cases, complex treatment is simply less tedious than a real treatment (eg. Fourier analysis)

A. Basic Properties

1. Complex number is ordered pair of real numbers (a, b)

2. Complex variable $z = (x, y) = x + iy$

3. Imaginary Unit: $i \equiv (0, 1)$

4. Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

5. Multiplication: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

where $\boxed{i^2 = -1}$

6. Complex Conjugate: $z^* = x - iy$ (conjugate of $z = x + iy$)

a. $zz^* = (x+iy)(x-iy) = x^2 + y^2$ is real!

b. To obtain complex conjugate, just change sign of i everywhere!

III A. (Continued)

Multiply complex conjugate of denominator

7. Division: $\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}\frac{(x_2 - iy_2)}{(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$

B. Functions in the Complex Domain

1. For a function $f(z)$ represented as a power series, we can use the expansion with complex z as the expansion variable.

2.

a. $e^{iz} = 1 + iz + \frac{1}{2!}(iz)^2 + \frac{1}{3!}(iz)^3 + \frac{1}{4!}(iz)^4 + \dots$

$$= \underbrace{\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right]}_{=\cos z} + i\underbrace{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right]}_{=\sin z}$$

$$= \cos z + i\sin z$$

b. NOTE: We can regroup terms because series is absolutely convergent for all z (d'Alembert ratio test succeeds).

c. Thus $\boxed{e^{iz} = \cos z + i\sin z}$ Valid for all complex z, but often used when z is real in Fourier analysis.

3. Separating real and imaginary parts of a complex function:

a. Any function $f(z)$ of a complex $z = x + iy$ may be written

$$f(z) = U(x,y) + iV(x,y)$$ where $U(x,y)$, $V(x,y)$ are real functions

b. Ex: $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy)$

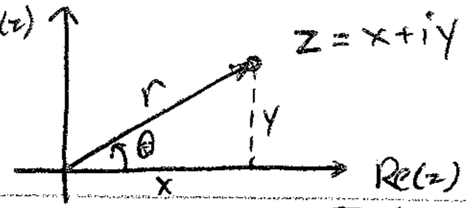
$U(x,y) = x^2 - y^2$ $V(x,y) = 2xy$

c. Notation: $Re[f(z)] = U(x,y) = x^2 - y^2$

$Im[f(z)] = V(x,y) = 2xy$

C. Polar Representation:

(Argand Diagram/Complex Plane)



1. $x = r\cos\theta \quad y = r\sin\theta \quad \Leftrightarrow \quad r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

(Cartesian) (Polar)

2. $\boxed{x+iy = r\cos\theta + ir\sin\theta = re^{i\theta}}$ Extremely valuable!

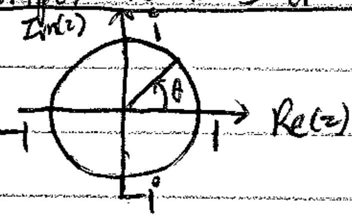
3. Multiplication & Division are easier in polar coordinates:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}$$

D. Complex Numbers of Unit Magnitude

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

1.



a. $e^{i\frac{\pi}{2}} = i$

b. $e^{i\pi} = -1$

c. $e^{i\frac{3\pi}{2}} = -i$

d. $e^{i2\pi n} = 1$ for integer n

E. Circular and Hyperbolic Functions

1. $\boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}}$ $\boxed{\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$

2. $\cosh iz = \cos z \qquad \sinh iz = i\sin z$

3. Since $(e^{i\theta})^n = e^{in\theta}$, we have $\boxed{\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n}$

↳ de Moivre's Theorem

F. Powers and Roots

1. $z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$

2. Roots: a. $z = re^{i(\theta+2m\pi)} \Rightarrow z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2m\pi}{n}\right)}$

b. In general $z^{\frac{1}{n}}$ is n-valued!

c. Ex: $1^{\frac{1}{3}} = e^{i\frac{\theta+2m\pi}{3}}$ 3 roots: $1, \frac{1}{2}(-1 \pm i\sqrt{3})$