Lecture #21: Laplace, Poisson, Wave, and Diffusion Equations

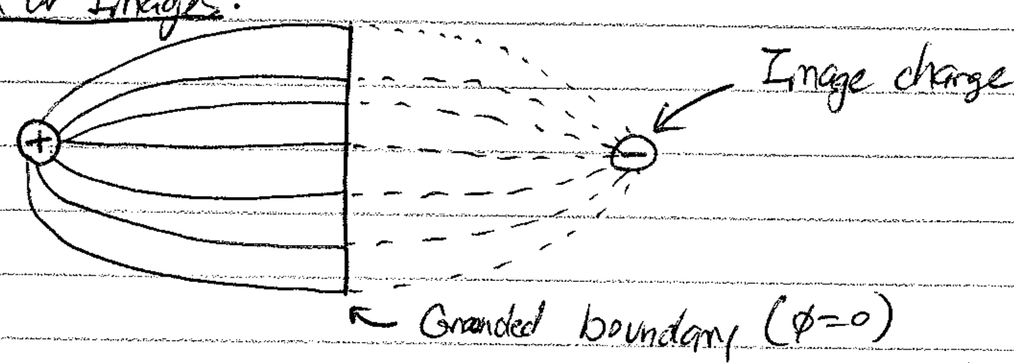
I. Laplace and Poisson Equations

A. General Properties

1. Laplace equation is prototypical elliptic PDE, $\nabla^2\psi = 0$.
2. Properties of the equation and its solutions are independent of the coordinate system.

3. Absence of Extrema:
   a. $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$ ← To sum to zero, second derivatives cannot all have same sign!
   b. Thus, stationary points (where $\nabla\psi = 0$) cannot be maxima or minima, but must be saddle points
   c. Ex: Electrostatics: $\nabla^2\phi = 0$ is solution of potential in charge-free regions.
      i. Potential cannot have an extremum where there is no charge.
      ii. Extrema must be on the boundary.

4. Uniqueness Theorem
   a. The solution to either Laplace or Poisson equation, subject to Dirichlet or Neumann boundary conditions, is unique.
   b. Method of Images:



      i. Two-charge system yields same potential ($\phi=0$) along boundary, so the solution for ϕ in region of interest is the same as for the original problem.

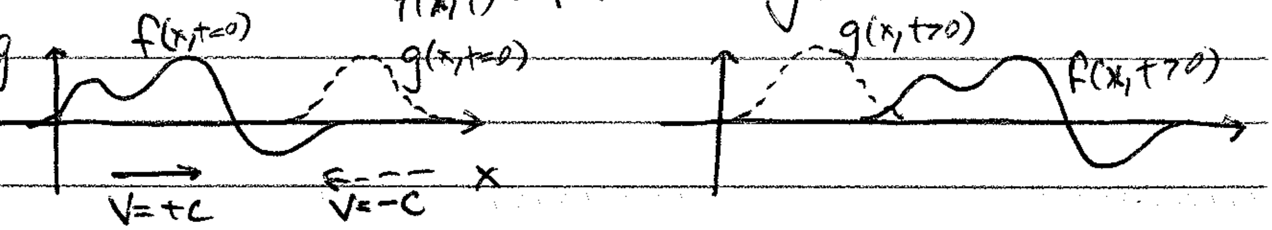
Lecture #21: (Continued)

II. Wave Equation

A. Traveling and Standing Waves

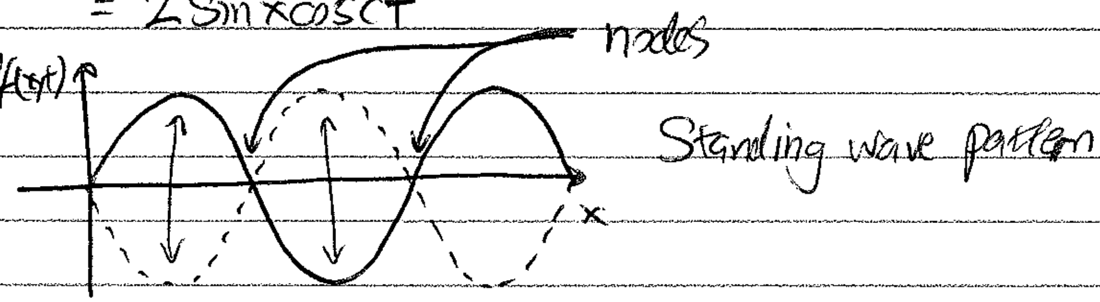
1. The wave equation is prototypical hyperbolic PDE, $\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}$

2. Two real characteristics along $x-ct=\text{constant}$ and $x+ct=\text{constant}$.
   a. General Solution $\psi(x,t) = f(x-ct) + g(x+ct)$
   b.



   c. f and g are completely arbitrary.

3. Standing Waves
   a. Consider $f=\sin(x-ct)$ and $g=\sin(x+ct)$
   b. $\psi(x,t) = (\sin x \cos ct - \cancel{\cos x \sin ct}) + (\sin x \cos ct + \cancel{\cos x \sin ct})$
   $= 2 \sin x \cos ct$
   c.



4. Solution using Separation of Variables
   a. $\psi(x,t) = X(x)\,T(t)$ ← yields standing wave solutions:
   $\psi_1 = \sin x \cos ct$ and $\psi_2 = \cos x \sin ct$
   b. But, $\psi_1 - \psi_2 = \sin x \cos ct - \cos x \sin ct = \sin(x-ct)$, so traveling wave solutions may be generated by linear combination.
   c. Total solution is the same for either standing or traveling wave bases.

II. (Continued)

B. d'Alembert's Solution

1. Solution given initial conditions for $\psi(x,t)$:

$$\psi(x,t=0) \quad \text{and} \quad \frac{\partial\psi}{\partial t}(x,t=0)$$

2. Take $\psi(x,0) = f(x) + g(x)$

$$\frac{\partial\psi}{\partial t}(x,0) = -cf'(x) + cg'(x)$$

3.

$$\text{(A)} \quad \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{\partial\psi(x,0)}{\partial t}dx = \frac{1}{2c}\Big[-cf(x)\Big]_{x-ct}^{x+ct} + \frac{1}{2c}\Big[cg(x)\Big]_{x-ct}^{x+ct}$$

$$= \frac{1}{2}\Big[\cancel{-f(x+ct)} + f(x-ct) + g(x+ct) \cancel{-g(x-ct)}\Big]$$

4. Using $\psi(x,t=0)$, we can write

$$\text{(B)} \quad \frac{1}{2}\Big[\psi(x+ct,0) + \psi(x-ct,0)\Big] = \frac{1}{2}\Big[\cancel{f(x+ct)} + g(x+ct) + f(x-ct) + \cancel{g(x-ct)}\Big]$$

5. Adding (A) + (B) and using $\psi(x,t) = f(x-ct) + g(x+ct)$, we obtain

$$\boxed{\psi(x,t) = \frac{1}{2}\Big[\psi(x+ct,0) + \psi(x-ct,0)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{\partial\psi(x,0)}{\partial t}dx}$$ d'Alembert's Solution

III. Diffusion Equation

A. Satisfying Boundary Conditions by Sum of Solutions

1. Diffusion equation is prototypical parabolic PDE, $\frac{\partial\psi}{\partial t} = a^2\frac{\partial^2\psi}{\partial x^2}$

2. Handling anisotropic diffusion: $\frac{\partial\psi}{\partial t} = a^2\frac{\partial^2\psi}{\partial x^2} + b^2\frac{\partial^2\psi}{\partial y^2} + c^2\frac{\partial^2\psi}{\partial z^2}$

a. Simply re-scale coordinates, $x' = \frac{x}{a}$, $y' = \frac{y}{b}$, $z' = \frac{z}{c}$, yielding

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x'^2} + \frac{\partial^2\psi}{\partial y'^2} + \frac{\partial^2\psi}{\partial z'^2}$$

III A (Continued) Hawes (4)

3. Consider 1D diffusion eq: $\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2}$

a. Separate variables: $\psi = T(t) X(x)$

i. $\frac{1}{T}\frac{dT}{dt} = -\alpha^2 a^2 = \frac{a^2}{X}\frac{d^2X}{dx^2} \Rightarrow T = e^{-\alpha^2 a^2 t}$ ← Needs $T \to 0$ as $t \to \infty$

$X = e^{\pm i \alpha^2 x}$

b. Solution: $\psi(x,t) = (\cos(\alpha x) \pm i \sin(\alpha x)) e^{-\alpha^2 t}$

c. NOTE: If $\alpha = 0$, we must include $\psi(x,t) = C_0' x + C_0$ as part of general solution.

d. Thus, General Solution: $\boxed{\psi(x,t) = (A\cos\alpha x + B\sin\alpha x) e^{-\alpha^2 a^4} + C_0' x + C_0}$

e. We may now choose values of A, B, C_0', C_0, and α to satisfy boundary conditions.

4. Apply BCs:

a. Finite Rod Length $\boxed{\psi(x,t) = \sum_n (A_n \cos\alpha_n x + B_n \sin\alpha_n x) e^{-\alpha_n^2 a^2 t} + C_0' x + C_0}$,

where α_n has discrete values that satisfy BCs.

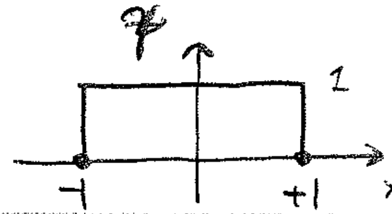
b. Infinite Rod: $\boxed{\psi(x,t) = \int (A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x) e^{-\alpha^2 a^2 t} d\alpha + C_0}$

Where $C_0' = 0$ to avoid divergence as $x \to \infty$.

5. General Principle:

Forming linear combinations of solutions with different parameters (α here) is a general way for adapting PDE solutions to specific boundary conditions.

III A. (Continued) Hawes ⑤



6. Ex: Warm Rod between cold reservoirs

a. $\psi = 0$ at $x = \pm 1$ at all times (cold reservoir).

b. Using general solution above for finite length rod,

$$\psi(x,t) = \sum_n \left(A_n \cos\alpha_n x + B_n \sin\alpha_n x\right) e^{-\alpha_n^2 a^2 t} + C_0' x + C_0$$

c. As $t \to \infty$, $\psi(x,t) \to 0$, so $C_0' = C_0 = 0$.

d. Problem is even in x, so $B_n = 0$.

e. To satisfy B.C.'s at $x = \pm 1$, $\alpha_n = \frac{n\pi}{2}$, so

$$\psi(x,t) = \sum_n A_n \cos\left(\frac{n\pi x}{2}\right) e^{-\frac{n^2\pi^2}{4} a^2 t}, \quad -1 < x < 1$$

f. Now, choose A_n to satisfy $\psi(x, t=0) = 1$ on $-1 < x < 1$.

i. $\sum_n A_n \cos\left(\frac{n\pi x}{2}\right) = 1$

ii. Multiplying by $\cos\left(\frac{n'\pi x}{2}\right)$ and integrating $\int_{-1}^{1} dx$, we obtain

$$A_n \underbrace{\int_{-1}^{1} \cos\left(\frac{n'\pi x}{2}\right)\cos\left(\frac{n\pi x}{2}\right) dx}_{\doteq\, \delta_{n'n}} = \int_{-1}^{1} (1) \cos\left(\frac{n'\pi x}{2}\right) dx = \frac{2}{n'\pi} \sin\left(\frac{n'\pi x}{2}\right)\Big|_{-1}^{1}$$

iii. Thus $A_{n'} = \frac{4}{n'\pi} \sin\left(\frac{n'\pi}{2}\right) \Rightarrow A_n = \frac{(-1)^m 4}{(2m+1)\pi}$ where $n = 2m+1$

g. Final Solution:

$$\boxed{\psi(x,t) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \cos\left[\frac{(2m+1)\pi x}{2}\right] e^{-\left[\frac{(2m+1)^2 \pi^2 a^2}{4}\right] t}}$$

7. 3D Diffusion Equation $a^2 \nabla^2 \psi = \frac{\partial \psi}{\partial t}$

a. Let $\psi = f(x,y,z)\, T(t)$

b. Separating variables, $\frac{1}{T}\frac{\partial T}{\partial t} = -k^2 \Rightarrow T = e^{-k^2 t}$

c. Helmholtz Eq: $\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + k^2 f = 0}$

III. (Continued)

B. Alternate Solutions

1. For 1D diffusion equation, $\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2}$, we seek solutions of the form, $\psi(x,t) = U(x/\sqrt{t})$

2. Let $\xi = \frac{x}{\sqrt{t}}$ for $U(\xi) = \psi(x,t)$;

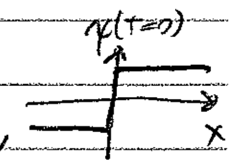
a. $\frac{\partial \psi}{\partial x} = \frac{U'}{\sqrt{t}}$ $\qquad \frac{\partial^2 \psi}{\partial x^2} = \frac{U''}{t}$ $\qquad \frac{\partial \psi}{\partial t} = -\frac{x}{2t^{3/2}} U'$

b. Thus, $\boxed{2a^2 U'' + \xi U' = 0}$ ODE for $U(\xi)$

3. Separating variables and integrating twice, we obtain

a. $U(\xi) = C_1 \int_0^{\xi} e^{-\xi'^2/4a^2} d\xi' + C_0$

b. Applying initial conditions $\psi(x, t=0) = \begin{cases} +1 & x > 0 \\ -1 & x < 0, \end{cases}$



$$\boxed{\psi(x,t) = \frac{1}{a\sqrt{\pi}} \int_0^{x/\sqrt{t}} e^{-\xi^2/4a^2} d\xi = \mathrm{erf}\left(\frac{x}{2a\sqrt{t}}\right)}$$ Gauss' Error Function

4. Generation of New Solutions by Differentiation

a. For diffusion equation with constant coefficients, we may generate new solutions by differentiating solution.

b. $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial t}$ are also solutions (differentiations commute).

c. Differentiate error function solution,

$$\psi_1(x,t) = \frac{1}{a\sqrt{\pi t}} e^{-x^2/(4a^2 t)}$$

$$\psi_2(x,t) = \frac{x}{2a^3\sqrt{t^3\pi}} e^{-\frac{x^2}{4a^2 t}}$$

5. Translate solution $\psi_1(x,t) \to \psi_1(x-\alpha, t)$ and integrate over α.

a. $\psi(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi_0(x - 2a\xi\sqrt{t}) e^{-\xi^2} d\xi$ where $\xi = \frac{x-\alpha}{2a\sqrt{t}}$.

Initial condition.

III. (Continued)

C. Other Geometries

1. Spherically symmetric heat flow, $u(r,t)$: $\frac{\partial u}{\partial t} = a^2\nabla^2 u = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$

a. Substitute: $u = \frac{v(r,t)}{r}$ to transform diffusion equation to

$$\boxed{\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial r^2}}$$

2. Cylindrical symmetry, $u(\rho,t)$: $\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}\right)$

a. Define $r \equiv \frac{\rho}{\sqrt{t}}$, where $u(\rho,t) = v(r)$

b. Lead's to ODE $\boxed{a^2 v'' + \left(\frac{a^2}{r} + \frac{r}{2}\right)v' = 0}$

c. We may separate variables and integrate to obtain

$$v(r) = \frac{c}{r}e^{-r^2/4a^2} = \frac{c\sqrt{t}}{\rho}e^{-\frac{\rho^2}{4a^2t}}.$$