Lecture #22 Green's Functions

I. Green's Functions

A. Basic Concept

1. Used to solve inhomogeneous differential equations, where the inhomogeneous term acts as a source, and you integrate over the source to obtain solution.

2. Ex: Poisson Equation: $-\nabla^2 \varphi(\underset{\sim}{r}) = \frac{1}{\epsilon_0}\rho(\underset{\sim}{r})$

a. Solution: $\varphi(\underset{\sim}{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\underset{\sim}{r}' \frac{\rho(\underset{\sim}{r}')}{|\underset{\sim}{r}-\underset{\sim}{r}'|}$

b. Integral contributes over entire region where $\rho(\underset{\sim}{r}')\neq 0$.

c. Right-hand side is an integral operator

$\Rightarrow$ Converts charge density $\rho(\underset{\sim}{r})$ to potential $\varphi(\underset{\sim}{r})$.

d. Green's Function: $\boxed{G(\underset{\sim}{r},\underset{\sim}{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}}$

i) This is the kernel that multiplies the inhomogeneous source term.

ii) Function of two variables $(\underset{\sim}{r},\underset{\sim}{r}')$: $\underset{\sim}{r}$ is constant, $\underset{\sim}{r}'$ is integrated.

e. Therefore $\varphi(\underset{\sim}{r}) = \int d^3\underset{\sim}{r}'\, G(\underset{\sim}{r},\underset{\sim}{r}')\,\rho(\underset{\sim}{r}')$

3. For more general problems, Green's function may depend on

a. Boundary Conditions

b. Form of differential operator.

4. KEY CONCEPT: a. $G(\underset{\sim}{r},\underset{\sim}{r}')$ gives contribution to φ at point $\underset{\sim}{r}$ due to a unit-magnitude source (delta function) at $\underset{\sim}{r}'$

b. We can determine φ at $\underset{\sim}{r}$ by summing (integrating) over all points of the source because the differential operator is linear

I. A 4. (Continued) Howes ②

c. Ex: i. Potential due to charge q at $\underset{\sim}{r}'$ $\phi(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{|r-r'|}$

ii. If I have a collection of charges q_j, then

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j}{|x - r_j'|}$$

iii. The continuum limit yields $\phi(\underset{\sim}{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\underset{\sim}{r}' \frac{\rho(\underset{\sim}{r}')}{|\underset{\sim}{x}-\underset{\sim}{r}'|}$

where charge density is $\int d^3\underset{\sim}{r}'\rho(\underset{\sim}{r}) = \underset{\substack{\uparrow \\ \text{total charge}}}{Q} = \sum_j q_j$

II. Green's Functions in 1D

A. Fundamental Definition

1. Consider a self-adjoint, inhomogeneous ODE $\mathcal{L}y = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y = f(x)$

   a. Interval $a \le x \le b$

   b. Boundary Conditions: Homogeneous at $x=a$ & $x=b$ (Meaning if y_1 satisfies BC's, so does Cy_1)

2. Defining Equation for G: $\boxed{\mathcal{L}G(x,x') = \delta(x-x')}$

3. Solution for $y(x)$: $\boxed{y(x) = \int_a^b G(x,x')\,f(x')\,dx'}$

   a. Check: $\mathcal{L}y(x) = \int_a^b \underbrace{\mathcal{L}G(x,x')}_{=\delta(x-x')} f(x')dx' = \int_a^b \delta(x-x')f(x')dx' = f(x).$

B. General Properties

Note: x is constant, but x' is variable that is differentiated.

1. Take $\mathcal{L}(x')\,G(x,x') = \delta(x-x')$ and integrate $\int_{x-\varepsilon}^{x+\varepsilon} dx'$

   a. $\int_{x-\varepsilon}^{x+\varepsilon} \mathcal{L}(x')G(x,x')\,dx' = \int_{x-\varepsilon}^{x+\varepsilon}\delta(x-x')dx' = 1$

   b. $\int_{x-\varepsilon}^{x+\varepsilon}\frac{d}{dx'}\left[p(x')\frac{dG(x,x')}{dx'}\right]dx' + \int_{x-\varepsilon}^{x+\varepsilon} q(x')\,G(x,x')\,dx' = 1$

II. B. (Continued)

2. First integral is exact $\Rightarrow$ $p(x')\dfrac{dG(x,x')}{dx'}\Big|_{x-\varepsilon}^{x+\varepsilon}$

3. <u>Necessary Discontinuity</u>:

$$p(x')\frac{dG(x,x')}{dx'}\Big|_{x-\varepsilon}^{x+\varepsilon} + \int_{x-\varepsilon}^{x+\varepsilon} q(x')G(x,x')dx' = 1$$

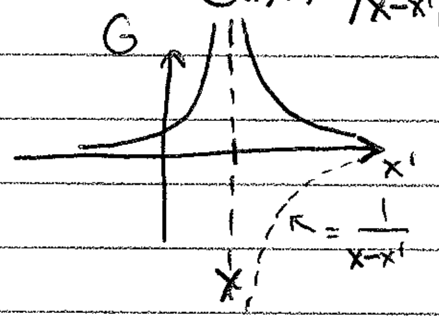
a. In $\lim_{\varepsilon\to 0^+}$, this <u>cannot</u> be satisfied if both $\dfrac{dG(x,x')}{dx'}$ and $G(x,x')$ are continuous.

b. We require $G(x,x')$ continuous, but allow discontinuity in $\dfrac{dG(x,x')}{dx'}$

c. In $\lim_{\varepsilon\to 0^+}$, integral vanishes, and we obtain

$$\lim_{\varepsilon\to 0^+}\left[\frac{dG(x,x')}{dx'}\Big|_{x'=x+\varepsilon} - \frac{dG(x,x')}{dx'}\Big|_{x'=x-\varepsilon}\right] = \frac{1}{p(x)}$$

d. <u>Ex</u>: 1-D $G(x,x') = \dfrac{1}{|x-x'|}$



$$\frac{dG(x,x')}{dx'} = \begin{cases} \dfrac{1}{(x-x')^2} & x'<x \\ \dfrac{-1}{(x-x')^2} & x'>x \end{cases}$$

$\Rightarrow$ Discontinuous slope.

4. <u>NOTE</u>: a. The key to understanding Green's functions is to be very careful handling it's <u>two</u> variables

b. One variable is always a constant, while the other varies (and is the one over which you integrate).

5. <u>Eigenfunction expansions in two variables</u>:

a. We are going to expand $G(x,x')$ in the orthonormal eigenfunctions of the Hermitian operator $\mathcal{L}$, ϕ_i

b. <u>NOTE</u>: Both x and x' are separately expanded in the ϕ_i

II. B.5. (Continued)

c. To expand a function $f(x,x')$ in ϕ_i

i. First, expand in $\phi_i(x)$ with $x' =$ constant

$$f(x,x') = \sum_n a_n(x')\,\phi_n(x) \quad \text{, where } a_n(x') = \langle \phi_n(x) | f(x,x') \rangle$$

Here $\langle \phi_n(x) | f(x,x') \rangle = \int_a^b \phi_n^*(x)\, f(x,x')\,dx = a_n(x')$ with $x' =$ const.

ii. Second, expand $a_n(x')$ in $\phi_m(x')$ with $x =$ constant

$$a_n(x') = \sum_m b_m\,\phi_m(x') \quad \text{where } b_m = \langle \phi_m(x') | a_n(x') \rangle$$

iii. Thus, $b_m = \langle \phi_m(x') | a_n(x') \rangle = \int_a^b \phi_m^*(x')\, a_n(x')\,dx'$

$$= \boxed{\int_a^b \phi_m^*(x') \boxed{\int_a^b \phi_n^*(x)\, f(x,x')\,dx}\, dx' \equiv C_{nm}}$$

iv. Finally $\boxed{f(x,x') = \sum_{n,m} C_{nm}\,\phi_n(x)\,\phi_m(x')}$

6. Now, let's expand $G(x,x')$ in terms of eigenfunctions of $\mathcal{L}$

a. First, expand $\delta(x-x')$ in terms of $\phi_i(x)$

i. $\delta(x-x') = \sum_n c_n(x')\,\phi_n(x)$

ii. $c_n(x') = \langle \phi_n(x) | \delta(x-x') \rangle = \int_a^b \phi_n^*(x)\,\delta(x-x')\,dx = \phi_n^*(x')$

iii. Thus $\boxed{\delta(x-t) = \sum_n \phi_n(x)\,\phi_n^*(x')}$

b. Now, $G(x,x') = \sum_{nm} g_{nm}\,\phi_n(x)\,\phi_m^*(x') \leftarrow$ Take $\phi_m^*(x')$, instead of $\phi_m(x')$, to match $\delta(x-x')$.

c. Use defining eq. for G: $\mathcal{L}(x) G(x,x') = \delta(x-x')$ and substitute.

i. NOTE: Here $\mathcal{L}(x)$ is operating on x, not x'!

ii. $\mathcal{L}(x) \sum_{nm} g_{nm}\,\phi_n(x)\,\phi_m^*(x') = \sum_m \phi_m(x)\,\phi_m^*(x')$

iii. $\sum_{nm} g_{nm}\,\lambda_n\,\phi_n(x)\,\phi_m^*(x') = \sum_m \phi_m(x)\,\phi_m^*(x')$

d. Take scalar product with $\phi^*_{m'}(x')$

$$\sum_{nm} g_{nm}\,\lambda_n\,\phi_n(x)\underbrace{\langle\phi^*_{m'}(x')|\phi^*_m(x')\rangle}_{\delta_{mm'}} = \sum_m \phi_m(x)\underbrace{\langle\phi^*_{m'}(x')|\phi^*_m(x')\rangle}_{=\delta_{mm'}}$$

e. Take scalar product with $\phi_{n'}(x)$

$$\sum_n g_{nm'}\,\lambda_n\underbrace{\langle\phi_{n'}(x)|\phi_n(x)\rangle}_{\delta_{n'n}} = \underbrace{\langle\phi_{n'}(x)|\phi_{m'}(x)\rangle}_{\delta_{n'm'}}$$

f. Thus, we obtain $g_{n'm'}\lambda_{n'} = \delta_{n'm'}$ or $\boxed{g_{nm} = \dfrac{\delta_{nm}}{\lambda_n}}$

g. Plugging back into expansion for $G(x,x')$: $\boxed{G(x,x') = \sum_n \dfrac{\phi_n(x)\,\phi^*_n(x')}{\lambda_n}}$

7. Symmetry of $G(x,x')$: $\boxed{G(x,x') = G^*(x',x)}$

a. This key property will enable us to establish a procedure for computing Green's Functions.

C. General Solution for $G(x,x')$

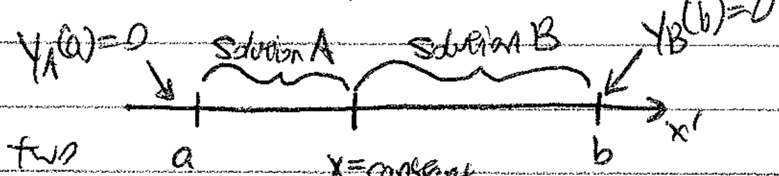
1. Consider a Hermitian operator $\mathcal{L}$ with homogeneous BCs on (a,b).

$$\boxed{\mathcal{L}y = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = f(x)}$$

Source in inhomogeneous ODE.

2. We will determine $G(x,x')$ in terms of solutions $\boxed{y_1(x),\ y_2(x)}$ of the homogeneous equation $\mathcal{L}y = 0$.

3. Strategy:



a. Split range $a \le x' \le b$ into two sections, $a \le x' < x$ and $x < x' \le b$, where x is constant.

b. In each half, satisfy B.C.'s: $y_A(a) = 0$, $y_B(b) = 0$!

II. C. (Continued)

4. The general homogeneous solution $\boxed{y(x') = C_1 y_1(x') + C_2 y_2(x')}$

5. Interval: $a \le x' < x$:

a. Choose C_1 & C_2 such that $y_A(a) = C_1 y_1(a) + C_2 y_2(a) = 0$

b. The most general form may have a constant $C_A(x) \leftarrow$ can be function of x.

c. Thus $G(x, x') = C_A(x)\, y_A(x')$, $a \le x' < x$

6. Following same procedure, $G(x, x') = C_B(x)\, y_B(x')$, $x < x' \le b$

7. Thus $\boxed{G(x,x') = \begin{cases} C_A(x)\, y_A(x') & a \le x' < x \\ C_B(x)\, y_B(x') & x < x' \le b \end{cases}}$

8. Apply Symmetry Condition $G(x, x') = G^*(x', x)$

a. Solution is consistent only if $C_A^*(x) = A\, y_B(x)$ and $C_B^*(x) = A\, y_A(x)$, where A is a constant

b. If we assume y_A and y_B can be chosen to be real, we obtain

$$\boxed{G(x, x') = \begin{cases} A\, y_A(x')\, y_B(x) & a \le x' < x \\ A\, y_B(x')\, y_A(x) & x < x' \le b \end{cases}}$$ Solution for $G(x, x')$

c. But, we still need to determine A.

9. Apply $\lim_{\varepsilon \to 0^+} \left[\frac{dG(x,x')}{dx'}\Big|_{\underbrace{x' = x+\varepsilon}_{x' > x}} - \frac{dG(x,x')}{dx'}\Big|_{\underbrace{x' = x-\varepsilon}_{x' < x}} \right] = \frac{1}{p(x)}$

a. $A\left[y_B'(x)\, y_A(x) - y_A'(x)\, y_B(x) \right] = \frac{1}{p(x)}$

b. Thus $\boxed{A = \left\{ p(x) \left[y_B'(x)\, y_A(x) - y_A'(x)\, y_B(x) \right] \right\}^{-1}}$ Solution for A

c. NOTES: i. A is actually a constant $\rightarrow$ does not depend on x!

ii. Term in brackets is Wronskian, $[\ldots] = W = \frac{C}{p(x)} \leftarrow$ can be shown.

II. (Continued)

D. <u>Example 1:</u>

1. Consider $-y'' = f(x)$, where $y(0) = y(1) = 0$.
2. General Solution to Homogeneous equation: $y'' = 0$

$$\boxed{y = c_0 + c_1 x}$$

3. Determine Solutions y_0 and y_1 that satisfy B.C's:
   a. $y(0) = 0 = c_0 + c_1(0) \Rightarrow c_0 = 0 \Rightarrow y_0(x) = x$
   b. $y(1) = 0 = c_0 + c_1(1) \Rightarrow c_1 = -c_0 \Rightarrow y_1(x) = 1 - x$
4. Compute coefficient A: $p(x) = -1$, $y_0'(x) = 1$ $y_1'(x) = -1$

$$A = \{p(x)[y_1'(x)y_0(x) - y_0'(x)y_1(x)]\}^{-1} = \{(-1)\underbrace{[(-1)(x) - (1)(1-x)]}_{\cancel{-x}-1+\cancel{x}}\}^{-1} = 1$$

5. Thus $\boxed{G(x,x') = \begin{cases} x'(1-x), & 0 \le x' < x \\ (1-x')x, & x < x' \le 1 \end{cases}}$

6. Thus, Solution $y(x)$ is $y(x) = \int_0^1 G(x,x')f(x')dx'$ for any $f(x')$.

7. a. For example, $f(x) = \sin(\pi x)$ yields

b. $y(x) = \int_0^1 G(x,x')\sin\pi x' dx' = \int_0^x x'(1-x)\sin\pi x' dx' + \int_x^1 (1-x')x \sin\pi x' dx'$

c. Using $\int_a^b x'\sin\pi x' dx' = \frac{\sin\pi x'}{\pi^2} - \frac{x'\cos\pi x'}{\pi}\Big|_a^b$, we obtain

$$= (1-x)\left[\frac{\sin\pi x'}{\pi^2} - \frac{x'\cos\pi x'}{\pi}\right]_0^x + x\left[\frac{-\cos\pi x'}{\pi}\right]_x^1 - x\left[\frac{\sin\pi x'}{\pi^2} - \frac{x'\cos\pi x'}{\pi}\right]_x^1$$

$$= (1-x)\left[\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi}\right] + x\left[\cancel{\frac{-(-1)}{\pi}} + \frac{\cos\pi x}{\pi}\right] - x\left[\cancel{\frac{-(1)(-1)}{\pi}} - \frac{\sin\pi x}{\pi^2} + \frac{x\cos\pi x}{\pi}\right]$$

$$= \frac{\sin\pi x}{\pi^2} - \cancel{\frac{x\cos\pi x}{\pi}} \cancel{- x\left(\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi}\right)} + \cancel{\frac{x\cos\pi x}{\pi}} \cancel{+ x\left(\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi}\right)}$$

$$\boxed{y(x) = \frac{\sin\pi x}{\pi^2}}$$

8. <u>NOTE:</u> With Green's functions, if you change $f(x)$, you need only repeat the integration of the Green's function Kernel with $f(x)$.

II. (Continued)

E. <u>Other Forms of Boundary Conditions</u>

1. Non-homogeneous boundary conditions: $y(a)=C_1 \quad y(b)=C_2$

a. Simply change y to $\quad U = y - \dfrac{C_1(b-x)+C_2(x-a)}{b-a}$

b. In terms of U, $U(a)=0$, $U(b)=0$.

2. <u>Initial Value Problems</u>: $\mathcal{L}_t y(t) = f(t) \qquad f(0)=0,\ f'(0)=0.$

a. Can construct $G(t,t')$ by invoking continuity of G at $t=t'$!

b. <u>BUT</u>, $G(t,t')$ is no longer symmetric.

3. <u>Ex:</u> Initial Value Problem: $\mathcal{L}y = \dfrac{d^2y}{dt^2} + y = f(t)$, $y(0)=0$, $y'(0)=0$

a. Homogeneous Solution: $\mathcal{L}y=0 \Rightarrow y_1 = \sin t$, $y_2 = \cos t$

b. <u>Key point</u>: No linear combination $y_0 = C_1y_1 + C_2y_2$ satisfies BCs at $t=0$, except for $C_1=C_2=0$.

i. Thus, for $t'<t$, we have $G(t,t')=0$.

c. <u>Second key point</u>: At $t'>t$, we have no constraining BC's, so

i. For $t'>t$, $G(t,t') = C_1(t)\sin t' + C_2(t)\cos t'$

d. Thus $\quad G(t,t') = \begin{cases} 0 & t'<t \\ C_1(t)\sin t' + C_2(t)\cos t' & t'>t \end{cases}$

e. Enforce continuity requirements:

i. $\displaystyle\lim_{\varepsilon\to 0^+}\Big[\underbrace{G(t,t+\varepsilon)}_{t'>t} - \underbrace{G(t,t-\varepsilon)}_{t'<t}\Big] = 0 \Rightarrow \boxed{C_1(t)\sin t + C_2(t)\cos t - 0 = 0}$

ii. $\displaystyle\lim_{\varepsilon\to 0^+}\Bigg[\underbrace{\frac{dG(t,t')}{dt'}\Bigg|_{t'=t+\varepsilon}}_{t'>t} - \underbrace{\frac{dG(t,t')}{dt'}\Bigg|_{t'=t-\varepsilon}}_{t'<t}\Bigg] = \frac{1}{p(t)}$ where $p(t)=1$ for $\mathcal{L}$.

iii. $\boxed{C_1(t)\cos t + C_2(t)(-\sin t) - 0 = 1}$

iv. Solving these equations yields $C_1(t)=\cos t$, $C_2(t)=\sin(t)$.

f. Thus $\cos t\sin t' - \sin t\cos t' = \sin(t'-t) \Rightarrow \boxed{G(t,t') = \begin{cases} 0 & t'<t \\ \sin(t'-t) & t'>t \end{cases}}$