Hawes ①

# Lecture #23 - Multidimensional Green's Functions and Probability

## I. Green's Functions in Multiple Dimensions

### A. Basic Properties

1. Many properties carry over from the 1D case.

2. Definition: $\boxed{\mathcal{L}\, G(\underline{r}, \underline{r}') = \delta(\underline{r} - \underline{r}')}$

   a. Here $\delta(\underline{r} - \underline{r}') = \delta(x - x')\,\delta(y - y')\,\delta(z - z')$

3. Hermitian 2nd-order PDE:

$$\mathcal{L}\, \psi(\underline{r}) = \nabla \cdot \left[ p(\underline{r})\, \nabla \psi(\underline{r}) \right] + q(\underline{r})\, \psi(\underline{r}) = f(\underline{r})$$

4. Solution to $\mathcal{L}\, \psi(\underline{r}) = f(\underline{r})$:

$$\psi(\underline{r}) = \int G(\underline{r}, \underline{r}')\, f(\underline{r}')\, d^3\underline{r}'$$

NOTE: $\underline{r}$ is held constant
$\underline{r}'$ is variable of integration.

5. If $\mathcal{L}\, \psi(\underline{r}) = \lambda \psi(\underline{r})$ defines a Hermitian eigenvalue problem with eigenfunctions $\phi_n(\underline{r})$ and eigenvalues $\lambda_n$, then

   a. Symmetric: $G(\underline{r}, \underline{r}') = G^*(\underline{r}', \underline{r})$

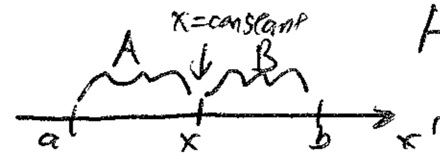
   b. Eigenfunction Expansion: $G(\underline{r}, \underline{r}') = \sum_n \dfrac{\phi_n^*(\underline{r}')\, \phi_n(\underline{r})}{\lambda_n}$

6. a. $G(\underline{r}, \underline{r}')$ is continuous and differentiable at all points $\underline{r} \neq \underline{r}'$.

   b. $G(\underline{r}, \underline{r}')$ has singularities in first derivatives so that second-order operator $\mathcal{L}$ generates the necessary delta function ((2) above).

I. (Continued)

B. Differences from 1D Case




1. Division into intervals $a < x' < x$, $x < x' < b$

   a. In the multi-dimensional case, one may not simply divide in to two intervals!

2. Specific forms of solution (such as solution for constant A) do not apply!

3. Ex: 3D Laplacian

   a. Obtain Green's Function for convenient BC's.

   b. Then, add solution to homogeneous equation $\mathcal{L}\,\psi(\underset{\sim}{r})=0$ to satisfy required BC's.

   c. $(\nabla')^2 G(\underset{\sim}{r},\underset{\sim}{r}') = \delta(\underset{\sim}{r}-\underset{\sim}{r}')$ where $\lim_{\underset{\sim}{r}'\to\infty} G(\underset{\sim}{r},\underset{\sim}{r}')=0$.

   operates on r' variable!

   d. NOTE: i) BCs are spherically symmetric and at an infinite distance from both $\underset{\sim}{r}$ and $\underset{\sim}{r}'$

   ii) We may make simplifying assumption $G(\underset{\sim}{r},\underset{\sim}{r}')$ is a function only of $\delta r = |\underset{\sim}{r}-\underset{\sim}{r}'|$

   iii) Then we can divide region along 1D variable $\delta r$.

   e. Following this strategy, we may obtain

$$\boxed{G(\underset{\sim}{r},\underset{\sim}{r}') = -\frac{1}{4\pi}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}}$$

   Fundamental Green's Function of Laplace's Equation in 3D

   f. May add a suitable solution to $\nabla^2\psi = 0$ (homogeneous eq.) to convert BC's from 0 at $\underset{\sim}{r}'=\infty$ to satisfy whatever BC's are needed.

# II. Probability

## A. Basic Concepts and Definitions

1. Random Event: Practically impossible to predict from initial sta
⇒ Includes when we have incomplete information about initial state.
Ex: Gas in a box: i. We do not know individual particle positions and velocities, but only average quantities such as pressure, temp.

2. Key Concept: Average properties of many similar events are predictable

3. Probability quantifies our level of ignorance!

4. Statistics connects observations on a small data sample to inferences about probable content of entire population

5. Def: Sample space: All possible mutually exclusive outcomes of an experiment.
   a. Mutually exclusive means if one event did occur, others did not.
   Ex: Flip a coin: If heads, then it cannot be tails.

6. Def: a. Trial: A single instance that produces an outcome.
   b. Event: Unique, equally likely occurrence
   c. Outcome: Events that satisfy some particular criterion.

7. Def: Experimental Probability

$$\boxed{P(x_i) \equiv \frac{\text{Number of times event } x_i \text{ occurs}}{\text{Total number of trials}}}$$

8. Def: Theoretical Probability

$$\boxed{P(x_i) \equiv \frac{\text{Number of outcomes } x_i}{\text{Total number of all possible events}}}$$

II. A. (Continued)

9. Experimental definition is appropriate when:
   a. Total number of possible events is not well defined
   b. Cannot identify equally likely outcomes.

10. Example: Two coin tosses: Theoretical Probability
   a. Outcome: Number of heads:
      i) Possible values $x_0, x_1, x_2$ for 0, 1, or 2 heads
   b. Possible Events:

Four possible events $N_{tot} = 4$

| Trial | Toss 1 | Toss 2 | Outcome |
| --- | --- | --- | --- |
| Result | H | H | $x_2$ |
| | H | T | $x_1$ |
| | T | H | $x_1$ |
| | T | T | $x_0$ |

   c. Theoretical Probability $P(x_0) = \frac{1}{4}$, $P(x_1) = \frac{2}{4} = \frac{1}{2}$, $P(x_2) = \frac{1}{4}$

11. Example: Grains of Sand: Experimental Probability
   a. Consider two piles of sand with same number of grains:
      one pile has black grains, other has white grains
   b. Mix both piles together thoroughly
   c. Counting all grains of sand is impractical
   d. Choosing a small sample, the probability of choosing a black grain is near $\frac{1}{2}$
   e. With a larger sample, the probability will get closer to $\frac{1}{2}$

12. Axioms
   a. $\underset{\text{impossibility}}{0} \leq P \leq \underset{\text{certainty}}{1}$
   b. Probabilities for mutually exclusive events add.
      Ex: One head from two coin tosses: $P = \frac{1}{2}$ where $P_1 = \frac{1}{4}$ for (T,H) & $P_2 = \frac{1}{4}$ for (H,T)

II. A. (Continued)

13. Or vs. "Exclusive Or"

a. In probability, "A or B" means A, B, or "both A and B".

b. Exclusive or (xor) means A or B but "not both A and B"

14. Example: Drawing Cards

a. What is probability for drawing a club or a jack?

b. NOTE: These events are not mutually exclusive, since there is a jack of clubs.

c. Deck of playing cards: 52 cards total → $N_{tot}$

4 suits of 13 cards each

d. Each card is equally likely to be drawn.

e. A: drawing a club: 13 events ← BUT → this include jack of clubs

B: drawing a jack: 4 events

f. Thus:

Theoretical Prob: $P = \underbrace{\frac{12}{52}}_{\text{clubs without jack}} + \underbrace{\frac{4}{52}}_{\text{all jacks}} = \frac{16}{52} = \boxed{\frac{4}{13} = P}$

B. Sets, Unions, Intersections

1. Consider a sample space S

2. Def: Subset, $\subset$

a. $A \subset S$ if all events in A are also in S.

3. Equality of A and B if $A \subset B$ and $B \subset A$

4. Def: Union, $\cup$

a. $A \cup B$ is all points in A, B, or both A and B.

5. Def: Intersection, $\cap$

a. $A \cap B$ is all points in both A & B.

6. If A and B have no common points, $A \cap B = \emptyset$ ← Empty Set.

7. <u>Subtraction</u>: $A - (A \cap B)$
   a. All points in A not also in B.

8. <u>Addition Rule for Probabilities</u>
   a. $P(A)$ is probability of event A from full sample S.
   b. $0 \leq P(A) \leq 1$
   c. $P(S) = 1$ Probability of entire sample space.
   d. $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$ Subtract double-counted points.
   e. If $A \cap B = \emptyset$ (Mutually exclusive outcomes), then
   $$P(A \cup B) = P(A) + P(B)$$

9. <u>Conditional Probability</u> $P(B|A)$
   a. <u>Def</u>: Probability that B occurs, given that A has already occurred.

   b. <u>Def</u>: <u>Ordered Events</u> $P(A, B)$
   Probability that A occurs, then B occurs.

   c. $\boxed{P(A,B) = \underbrace{P(A)}_{\text{Probability of A}} P(B|A)}$ Probability of B <u>after</u> A has occurred.

   d. Thus $\boxed{P(B|A) = \dfrac{P(A,B)}{P(A)}}$

10. <u>Example</u>: Conditional Probability
   a. Consider a box of 10 identical ~~reed~~ & 20 identical blue pens.
   b. What is $P(R,B)$? (Drawing red pen, then drawing blue pen).

II B. 10. (Continued) 

c. NOTE: Pens are not replaced after they are drawn;

d.

$$P(R,B) = \left(\frac{10}{30}\right) \cdot \left(\frac{20}{29}\right) = \boxed{\frac{20}{87}}$$

10 red, 20 blue; 30 total, 29 total

$P(A)$   $P(B|A)$

11 a. If A and B are independent, then $P(B|A) = P(B)$

b. Thus $P(A,B) = P(A)\,P(B)$

c. This would be the case if we replace pens after each drawing.

d. For A and B independent, we may write

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{where } P(A \cap B) = P(A)\,P(B)$$

12. Bayes Theorem: $P(A|B)\,P(B) = P(B|A)\,P(A)$

C. Counting Permutations (Ordered) and Combinations (Unordered)

1. Permutations: (Order Matters)

a. How many ways can we arrange (permute) n different letters?

$$\underline{n} \cdot \underline{n-1} \cdot \underline{n-2} \cdot \ldots \cdot \underline{2} \cdot \underline{1} = \boxed{n!}$$

b. How many (ordered) ways can we seat n people in k chairs? $(n > k)$

$$\underset{1}{\underline{n}} \cdot \underset{2}{\underline{(n-1)}} \cdot \underset{3}{\underline{(n-2)}} \ldots \underset{k-1}{\underline{(n-k)}} \cdot \underset{k}{\underline{(n-k+1)}} = \boxed{\frac{n!}{(n-k)!}}$$

2. Combinations: (Order does not matter)

a. How many ways can you choose k particles from n distinguishable particles?

b. NOTE: for the same k particles, there are k! possible permutations.

II. C2 (Continued)

c. Thus divide by possible permutations $\boxed{\dfrac{n!}{(n-k)!\,k!}}$

d. Symbol: $\binom{n}{k}$ "n choose k" $\boxed{\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}}$

e. This is the binomial coefficient.

3. Classical vs. Quantum Statistics (n particles, k possible states)

a. Maxwell-Boltzmann Statistics: Distinguishable particles

i. Total possibilities $\boxed{k^n}$

b. Bose-Einstein Statistics: Indistinguishable Particles

i. Wave function symmetric under particle interchange

ii. Total possibilities: $\boxed{\binom{n+k-1}{n} = \binom{n+k-1}{k-1}}$

c. Fermi-Dirac Statistics: Indistinguishable Particles

i. Wave function reverses sign under particle interchange.

ii. Total Possibilities $\boxed{\binom{k}{n}}$

iii. NOTE: Probability is zero if $n > k$ (more particles than states), since maximum one particle per state.