

Lecture #3: Derivatives, Integrals, and the Delta Function

I. Derivatives

A. Definitions and Properties

1. Def: Derivative $\frac{df}{dx}$ of $f(x)$

$$\frac{df(x)}{dx} = \lim_{\epsilon \rightarrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

2. Def: Differential $df = f(x+dx) - f(x) = \frac{df}{dx} dx$

a. Mean Value Theorem tells us to evaluate $\frac{df}{dx}$ at ξ where $x < \xi < x+\epsilon$.

3. Multidimensional Functions $f(x, y, z)$

a. $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

b. Def: Partial Derivative $\frac{\partial f(x, y, z)}{\partial x}$

i) Evaluated with variables y & z kept fixed

ii) If independence of variables is ambiguous (such as when only two of x, y, z are independent),

$$\left(\frac{\partial f}{\partial x}\right)_y \quad \text{or} \quad \left(\frac{\partial f}{\partial x}\right)_z$$

4. Def: Cross Derivatives (for higher derivatives)

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

independent of order of differentiation.

I. A. (Continued)

5. Def: Chain Rule

a. If $F(x, y, z)$ and $x(s), y(s), z(s)$, then

$$\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} + \frac{\partial f}{\partial z} \frac{dz}{ds}$$

6. Useful Formula: For a function $F(x, y)$, when $df=0$, we have

a. $df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0$

b. Solving for $\left(\frac{dy}{dx}\right)_f$,
Constant f
Since $df=0!$

$$\left(\frac{dy}{dx}\right)_f = - \frac{\left(\frac{\partial f}{\partial x}\right)_y}{\left(\frac{\partial f}{\partial y}\right)_x}$$

c. Trajectory in (x, y) plane where $F = \text{constant}$!

7. Ex: Lagrangian Mechanics

a. $L(x, \dot{x}, t)$ where $\dot{x} \equiv \frac{dx}{dt}$

b. $\frac{d}{dt} L(x, \dot{x}, t) = \left[\frac{\partial L}{\partial x} \dot{x} + \frac{\partial L}{\partial \dot{x}} \ddot{x} + \frac{\partial L}{\partial t} \right]$
↑ ordinary derivative ↑ partial derivatives

c. Similar manipulations in kinetic plasma physics

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = 0.$$

B. Stationary Points

1. Key idea for optimization. (Highly valuable skill!)
2. For a function of n variables, $F(x_1, x_2, \dots, x_n)$, one can specify a "direction" in $\{x_i\}$ space by $\frac{dx_1}{ds}, \frac{dx_2}{ds}, \dots, \frac{dx_n}{ds}$

I, B. (Continued)

3. A necessary (but not sufficient) condition to find a minimum of f

a. $\frac{df}{ds} = 0$

Def: Stationary Point

b. Equivalent to

$\frac{\partial f}{\partial x_i} = 0$ for $i=1, \dots, n$

4. Stationary Points:

a. Minimum

if

$\frac{\partial^2 f}{\partial x_i^2} > 0$ for $i=1, \dots, n$

b. Maximum

if

$\frac{\partial^2 f}{\partial x_i^2} < 0$ for $i=1, \dots, n$

c. Otherwise, Saddle Point.

5. Extremely valuable topic in Quantitative Financial Analysis.

II. Integrals

Evaluation of integrals is ubiquitous in solving physics problems, requiring skill, pattern recognition and a few tricks.

A. Integration by Parts

1. $\int_a^b U dv = UV|_a^b - \int_a^b v du$

2. Ex:

$\int_a^b x \sin x dx = (-x \cos x)|_a^b - \int_a^b (-\cos x) dx$

$U=x \quad dv=dx$

$v=-\cos x \quad dv=\sin x dx$

$+ \int_a^b \cos x dx = (\sin x)|_a^b$

$= a \cos a - b \cos b + \sin b - \sin a$

II. (Continued)

Haves ④

B. Identify a Special Function

1. Identifying a special function enables you to exploit the full body of knowledge on the properties and evaluation.
2. The key is to recognize the function and know where to look for it.
 - a. Schaum's Outline: Spiegel, Lipschutz, & Liu
 - b. Abramowitz and Stegun
 - c. Gradshteyn and Ryzhik

3. A few examples:

a. Error Functions: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ $\operatorname{erf}(\infty) = 1$

Very useful in Statistics →

$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$ $\operatorname{erfc}(\infty) = 1 - \operatorname{erf}(x)$

b. Gamma Function $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$

c. Factorial Integral $n! = \int_0^{\infty} t^n e^{-t} dt$ $n! = \Gamma(n+1)$

d. Riemann Zeta Functions $\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1} dt}{e^t - 1}$

e. Plus Bessel Functions, and more.

C. Other Methods

1. Contour Integration in complex plane (Math Methods II)

2. Integration or Differentiation with respect to a parameter.

a. Ex: $I = \int_0^{\infty} \frac{e^{-x^2}}{x^2 + a^2} dx \Rightarrow J(a) = \int_0^{\infty} \frac{e^{-t(x^2 + a^2)}}{x^2 + a^2} dx$

where $I = e^{a^2} J(1)$. (See text for full solution.)

3. Expand, then Integrate: $I = \int_0^1 \frac{dx}{x} \ln\left(\frac{1+x}{1-x}\right) = \int_0^1 dx \cdot 2 \left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \dots \right]$

↑
Easy to integrate power series!

II. C (Continued)

Howes 6

4. Convert Trigonometric Functions to Exponential Forms

a. Ex: $I = \int_0^{\infty} e^{-at} \cos bt \, dt$

b. Since $\cos bt = \operatorname{Re} [e^{ibt}]$, so $I = \operatorname{Re} \int_0^{\infty} e^{-(a+ib)t} \, dt$

c. $I = \operatorname{Re} \left[\frac{e^{-(a+ib)t}}{-a+ib} \Big|_0^{\infty} \right] = \operatorname{Re} \left[0 - \frac{1}{-a+ib} \right] = \operatorname{Re} \left[\frac{1}{a-ib} \right] = \frac{a}{a^2+b^2}$

5. Recursion

a. Using successive integration by parts to obtain recursion formula.

b. Ex: $I_n = \int_0^1 t^n \sin(\pi t) \, dt$ for positive integer n .

(i) Integrate by parts twice; taking $u = t^n$
 $dv = \sin(\pi t) \, dt$

$$\Rightarrow I_n = \frac{1}{\pi} - \frac{n(n-1)}{\pi^2} I_{n-2}$$

where $I_0 = \frac{2}{\pi}$, $I_1 = \frac{1}{\pi}$.

D. Multiple Integrals

1. For a function $f(x,y)$, we have

$$\iint f(x,y) \, dx \, dy \quad \text{or} \quad \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} f(x,y) \, dy \, dx$$

This form is much more explicit about order of integration.

2. Can also be written in a general way:

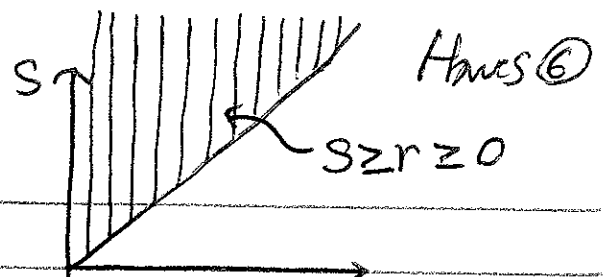
a. $\int_S f(x,y) \, dA$ ← unspecified: $\begin{cases} dA = dx \, dy \\ dA = r \, dr \, d\theta \end{cases}$

b. $\int_V f(x,y,z) \, dV$

3. Can change order of integration to evaluate some integrals more easily. a) But, be careful to determine limits of integration.

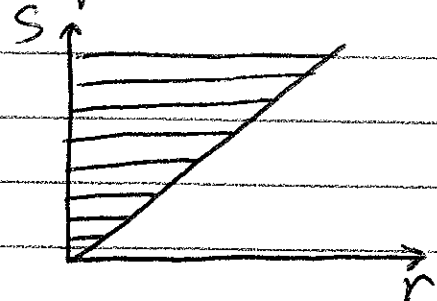
II, D. (Continued)

4. Ex: a. $I = \int_0^{\infty} e^{-r} dr \int_0^{\infty} \frac{e^{-s}}{s} ds$



b. Switch order, we cover same area in (r, s) plane:

$$I = \int_0^{\infty} \frac{e^{-s}}{s} ds \int_0^s e^{-r} dr$$



c. $\int_0^s e^{-r} dr = (1 - e^{-s})$, so

$$\int_0^{\infty} \frac{e^{-s}}{s} (1 - e^{-s}) ds$$

d. Express $1 - e^{-s} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} s^n$, thus $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_0^{\infty} ds s^{n-1} e^{-s}$

e. Thus $I = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} (n-1)! = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2!$ Factorial Integral = $(n-1)!$

5. Evaluation in Polar Coordinates

a. Use $dx dy = r dr d\theta$

b. Ex: i) To evaluate $I = \int_0^{\infty} e^{-x^2} dx$, take $I^2 = \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy$

ii) then $= \int_0^{\pi/2} d\theta \int_0^{\infty} r dr e^{-r^2} = \frac{\pi}{2} \int_0^{\infty} \frac{1}{2} du e^{-u} = \frac{\pi}{4}$
 $u = r^2 \quad du = 2r dr$

iii) Thus, $I = \frac{\sqrt{\pi}}{2} \Rightarrow$ Well known result

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

E. Changing Integration Variables

1. Single Integral: To change from x to $y = y(x)$

a) Replace dx by $\frac{dx}{dy} dy$

b) Update integration limits $\int_{x_1}^{x_2}$ to $\int_{y(x_1)}^{y(x_2)}$

2. Multiple Integrals

a. Changing variables for multiple integrals is much more complicated.

b. Def: Jacobian i) From (x,y) to (u,v)

$$dx dy = J du dv$$

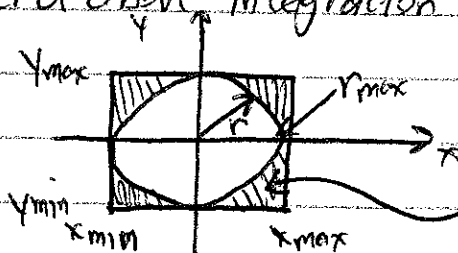
ii) where $J = \frac{\partial(x,y)}{\partial(u,v)}$ is Jacobian

c. Ex: From (x,y) to (r,θ) $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$, so

$$dx dy = r dr d\theta$$

d. General Methods for computing Jacobians in Chap 4.

e. Integration Limits: Unless limits are at infinity, one must be very careful about integration limits under transformation



Corner regions must be taken into account!

III. The Dirac Delta Function

A. Definition

1. Def: Dirac Delta Function a)

$$\delta(x) = 0 \quad x \neq 0$$

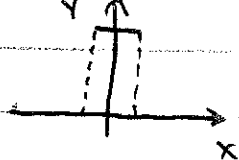
$$f(0) = \int_a^b f(x) \delta(x) dx$$

b) Normalization

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

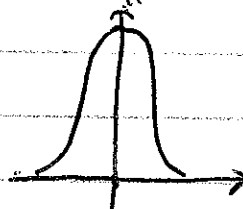
2. Limit of a Sequence of functions:

$$a) \delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$$



$$b) \delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

(Leads to Hermite Polynomials)



III. A.2. (Continued)

$$c) S_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}$$

$$d) S_n(x) = \frac{\sin nx}{\pi x}$$

Havees (8)

3. More formally, $\delta(x)$ is not a function but a distribution such that
$$\int_{-\infty}^{\infty} \delta(x) f(x) dx \equiv \lim_{n \rightarrow \infty} \int S_n(x) f(x) dx$$

B. Properties of $\delta(x)$

1. Even parity: $\delta(-x) = \delta(x)$

2. If $a > 0$, $\delta(ax) = \frac{1}{a} \delta(x)$

3. $\int_{-\infty}^{\infty} \delta(x-x_0) f(x) dx = f(x_0)$

4. For $g(x)$ has simple zeros on the real axis at $x=a_i$ (thus $g'(a_i) \neq 0$)

$$\delta(g(x)) = \sum_i \frac{\delta(x-a_i)}{|g'(a_i)|}$$

5. Derivative:

$$\int_{-\infty}^{\infty} f(x) \delta'(x-x_0) dx = -f'(x_0)$$

6. Three-Dimensional: $\delta(\mathbf{r}) = \delta(x) \delta(y) \delta(z)$

$$\int_V f(\mathbf{r}') \delta(\mathbf{r}'-\mathbf{r}_0) d\mathbf{r}' = f(\mathbf{r}_0)$$

7. Exponential Form:

$$\delta(t-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega$$

Very useful for
Fourier Transforms.

C. Discrete Version: Kronecker Delta

1.
$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Very useful for concisely
expressing and simplifying
calculations in summation
notation.