Lecture #4 Determinants

I. Determinants

Determinants are valuable in the solution of linear systems of equations.

A. Homogeneous Linear Equations

1. Consider a linear set of n homogeneous equations with n unknowns

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$
$$b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$$
$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$$

QUESTION: 2. Under what conditions is there a nontrivial solution? (Not trivial $x_1 = x_2 = x_3 = 0$)

3. Vector Notation a. If $\underset{\sim}{x} = (x_1, x_2, x_3)$, $\underset{\sim}{a} = (a_1, a_2, a_3)$, etc, then

$$\underset{\sim}{a} \cdot \underset{\sim}{x} = 0 \qquad \underset{\sim}{b} \cdot \underset{\sim}{x} = 0 \qquad \underset{\sim}{c} \cdot \underset{\sim}{x} = 0$$

b. Application: This occurs in many fields of physics, for example when solving for linear waves in a system (linear dispersion relation)

c. Geometrical Interpretation:

i) Vector $\underset{\sim}{x}$ is orthogonal to $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$!

ii) Volume spanned by $\underset{\sim}{a}$, $\underset{\sim}{b}$, & $\underset{\sim}{c}$ is given by triple scalar product (equal to the determinant)

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c} = \det(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) = D_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

ANSWER 4. $\boxed{\text{If } D \neq 0 \text{, then only the trivial solution } x_i = 0 \text{ exists!}}$

5. As we will see, if $D_3 = 0$, then to obtain a solution for x_1, x_2, & x_3 requires that you choose a value for one of the variables (say x_3) and solve for $x_1 = f_1(x_3)$, $x_2 = f_2(x_3)$

Functions will depend on coefficients $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$

I. (Continued) Howes ②

B. Inhomogeneous Linear Equations

1.
$$a_1 x_1 + a_2 x_2 = h_1$$
$$b_1 x_1 + b_2 x_2 = h_2$$

2a. In contrast to the homogeneous case ($h_1 = h_2 = 0$), the solution of the inhomogeneous will be uniquely determined (no free parameter) if a solution exists.

b. The solutions x_1 & x_2 will depend on h_1 & h_2 as well as $\underset{\sim}{a}$, $\underset{\sim}{b}$.

C. Definitions:

1. RC Order: For a 2D array, the nth row & mth column is (n, m)
(row, column) $\rightarrow$ (r, c)

2. Permutations: For n unique objects in some reference order, there are $n!$ possible permutations (different orders).
   a. $\underline{n} \cdot \underline{n-1} \cdot \underline{n-2} \cdot \ldots \cdot \underline{1} = n!$

3. Parity: a. Pairwise interchanges may be used to alter order.
$$abcd \Rightarrow adcb$$
b. The parity (even or odd) is the number of pairwise interchanges to achieve a permutation from a reference order

Ex: $abcd \Rightarrow \underset{\text{odd }(-1)}{adcb} \Rightarrow \underset{\text{even }(+1)}{dacb}$ Same parity regardless of path.

OR: $abcd \Rightarrow \underset{\text{odd}}{abdc} \Rightarrow \underset{\text{even}}{adbc} \Rightarrow \underset{\text{odd}}{dabc} \Rightarrow \underset{\text{even}}{dacb}$

4. Levi-Civita symbol: $\varepsilon_{ijk\ldots}$
   a. For an n-object system, $\varepsilon_{ijk\ldots}$ has n subscripts
   b.
$$\left.\begin{array}{ll} \varepsilon_{ijk\ldots} = +1 & \text{even permutation} \\ \varepsilon_{ijk\ldots} = -1 & \text{odd permutation} \end{array}\right\} \text{from a reference order.}$$
$\varepsilon_{ijk\ldots} = 0$ not a permutation (repeated index)

5. <u>Def:</u> <u>Determinant</u> for a $n \times n$ array (n equations, n unknowns)

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{vmatrix} = \sum_{ijk\ldots} \varepsilon_{ijk\ldots}\, a_{1i} a_{2j} \cdots$$

D. <u>Computing Determinants</u>

1. <u>Ex:</u> 2×2 Array:

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \overset{(+1)}{\varepsilon_{12}}\, a_{11} a_{12} + \overset{(-1)}{\varepsilon_{21}}\, a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21}$$

2. <u>Ex:</u> 3×3 Array:

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{ijk} \varepsilon_{ijk}\, a_{1i} a_{2j} a_{3k}$$

$$= \overset{\varepsilon_{123}}{a_{11} a_{22} a_{33}} - \overset{\varepsilon_{132}}{a_{11} a_{23} a_{32}} + \overset{\varepsilon_{312}}{a_{13} a_{21} a_{32}} - \overset{\varepsilon_{321}}{a_{13} a_{22} a_{31}} + \overset{\varepsilon_{231}}{a_{12} a_{23} a_{31}} - \overset{\varepsilon_{213}}{a_{12} a_{21} a_{33}}$$

3. <u>Shortcut for 2×2 & 3×3 Determinants:</u>

a.
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

(−1) (+1)

b.
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

I. (Continued)

E. <u>Properties of Determinants:</u>

1. Interchanging two rows (or columns) changes sign of D.

2. Transposition (all $b_{ji} = a_{ij}$) does not alter D.

3. Multiplying all members of a row (or column) by a factor k changes value to kD.

$$k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} ka_1 & a_2 \\ kb_1 & b_2 \end{vmatrix} = \begin{vmatrix} ka_1 & ka_2 \\ b_1 & b_2 \end{vmatrix}$$

4. If elements of a row (or column) are sums of two quantities, $D = D_1 + D_2$.

$$\begin{vmatrix} a_1 + x_1 & a_2 \\ b_1 + x_2 & b_2 \end{vmatrix} = \begin{vmatrix} x_1 & a_2 \\ x_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

5. <u>Implications:</u>

a. Any determinant with two rows (or columns) equal (or proportional) will have $D = 0$.

b. The value of D is unchanged if a multiple of one row (or column) is added element-by-element to another

c. If any row or column is all zeros, $D = 0$.

I. (Continued)

F. Laplacian Expansion by Minors

1. Def: Minor

a. For an order n determinant, the minor M_{ij} associated with element a_{ij} is the order $(n-1)$ determinant produced by removing the ith row and jth column.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Minor of a_{22} $\Rightarrow M_{22} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$

2. Expansion by Minors: Choose any row or column to expand,

$$\boxed{D_n = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}}$$

a. NOTE: Choosing a column with lots of zeros is best.

3. Ex: Expanding by minors, evaluate

$$D = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

a. Expand across top row!

$$D = \sum_{j=1}^{4} a_{1j} (-1)^{1+j} M_{1j}$$

$$= \underset{a_{11}}{(0)} M_{11} - \underset{a_{12}}{(1)} M_{12} + \underset{a_{13}}{(0)} M_{13} - \underset{a_{14}}{(0)} M_{14} = -M_{12}$$

b. $M_{12} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = -(-1)(1)(-1) = -1$

c. Thus $\boxed{D = -M_{12} = 1}$

I. (Continued)

G. Solving Systems of Linear Equations

1.
$$\begin{aligned} a_1 x_1 + a_2 x_2 + a_3 x_3 &= h_1 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 &= h_2 \\ c_1 x_1 + c_2 x_2 + c_3 x_3 &= h_3 \end{aligned} \quad \text{so } D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. Cramer's Rule:

a. Take
$$x_1 D = \begin{vmatrix} a_1 x_1 & a_2 & a_3 \\ b_1 x_1 & b_2 & b_3 \\ c_1 x_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x_1+a_2x_2+a_3x_3 & a_2 & a_3 \\ b_1x_1+b_2x_2+b_3x_3 & b_2 & b_3 \\ c_1x_1+c_2x_2+c_3x_3 & c_2 & c_3 \end{vmatrix}$$
$$\begin{pmatrix} a_1x_1+a_2x_2+a_3x_3 \\ b_1x_1+b_2x_2+b_3x_3 \\ c_1x_1+c_2x_2+c_3x_3 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

b. Thus
$$\boxed{x_1 = \frac{1}{D} \begin{vmatrix} h_1 & a_2 & a_3 \\ h_2 & b_2 & b_3 \\ h_3 & c_2 & c_3 \end{vmatrix}}$$

c. Similarly,
$$x_2 = \frac{1}{D} \begin{vmatrix} a_1 & h_1 & a_3 \\ b_1 & h_2 & b_3 \\ c_1 & h_3 & c_3 \end{vmatrix}, \text{ etc.}$$

For x_i solution, simply replace ith column with RHS values!

3. a. These solutions for x_i are unique.

b. If all $h_i = 0$, then unique solution for $D \neq 0$ is all $x_i = 0$.

H. Linearly Dependent Equations

1. For n linear equations with n variables,
   - if $D \neq 0$, equations are linearly independent
   - if $D = 0$, equations are linearly dependent

2. Linearly dependent equations arise often in physics.

a. For homogeneous equations (all $h_i = 0$), one or more equations in the set are linear combinations of the others. Thus, fewer than n equations for n unknowns

b. Thus, we may choose values for one (or more) variables, and we can solve for all other variables in terms of those values.

$\Rightarrow$ Thus, we obtain a <u>manifold</u> (a parameterized set) of solutions.

c. <u>Ex:</u> <u>Plasma Waves:</u>
The linear response in plasmas leads to wave behavior. Solution for linear plasma wave properties is often important for determining the physical response of a system to perturbations.

★★★ 3. $\boxed{\text{A system of } n \text{ homogeneous linear equations has non-trivial solutions only if } D=0.}$

4. The scale of the solutions (amplitude) is arbitrary.

5. <u>Ex:</u> <u>Linearly Dependent Homogeneous Equations</u>

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + 3x_2 + 5x_3 = 0$$
$$x_1 + 2x_2 + 3x_3 = 0$$

a. $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = (1\cdot3\cdot3) + (1\cdot5\cdot1) + (1\cdot1\cdot2) - (1\cdot3\cdot1) - (1\cdot1\cdot3) - (1\cdot5\cdot2)$

$= 9+5+2-3-3-10 = 0! \Rightarrow$ Nontrivial Solutions exist.

b. <u>Solving:</u> i) NOTE! 3rd equation is half the sum of the other two.
$\Rightarrow$ Drop it!

ii) ② − ① $= 2x_2 + 4x_3 = 0 \quad \Rightarrow x_2 = -2x_3$

iii) 3① − ② $= 2x_1 - 2x_3 = 0 \quad \Rightarrow x_1 = x_3$

iv) Thus $(x_1, x_2, x_3) = (x_3, -2x_3, x_3) = (1,-2,1)\, x_3$

The relationships between $x_1, x_2, \& x_3$, given by $(1,-2,1)$, has important physical significance for all solutions.

Arbitrary Scaling Factor.

I. Gauss Elimination:

1. A versatile and robust procedure for evaluating determinants, solving linear equations, and matrix inversion ⇒ NUMERICAL IMPLEMENTATION

2. Ex: Solve

$$3x \quad 2y + z = 11$$
$$2x + 3y + z = 13$$
$$x + y + 4z = 12$$

For numerical accuracy.

a. Conditioning: Arrange to have largest coefficients on diagonal.

b. Solving Procedure:

① Divide each row by initial coefficient

$(\times \frac{1}{3})$ $x + \frac{2}{3}y + \frac{1}{3}z = \frac{11}{3}$

$(\times \frac{1}{2})$ $x + \frac{3}{2}y + \frac{1}{2}z = \frac{13}{2}$

$x + y + 4z = 12$

② Subtract row 1 from 2 & 3:

$$x + \frac{2}{3}y + \frac{1}{3}z = \frac{11}{3}$$
$$\frac{5}{6}y + \frac{1}{6}z = \frac{17}{6}$$
$$\frac{1}{3}y + \frac{11}{3}z = \frac{25}{3}$$

③ Divide row 2 & 3 by initial coefficient

$x + \frac{2}{3}y + \frac{1}{3}z = \frac{11}{3}$

$(\times \frac{6}{5})$ $y + \frac{1}{5}z = \frac{17}{5}$

$(\times 3)$ $y + 11z = 25$

④ Subtract row 2 from 3:

$$x + \frac{2}{3}y + \frac{1}{3}z = \frac{11}{3}$$
$$y + \frac{1}{5}z = \frac{17}{5}$$
$$\frac{54}{5}z = \frac{108}{5}$$

⑤ Solve for z from row 3

$$\boxed{z = 2}$$

⑥ Solve for y from row 2 and z=2:

$$y + \frac{2}{5} = \frac{17}{5} \Rightarrow \boxed{y = 3}$$

⑦ Solve for x from row 1

$$x + 2 + \frac{2}{3} = \frac{11}{3} \Rightarrow \boxed{x = 1}$$

c. Find determinant: ① $D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix}$

② We operated by $(\frac{1}{3})(\frac{1}{2})(\frac{6}{5})(3)$, so

$$D = (3)(2)\left(\frac{5}{6}\right)\left(\frac{1}{3}\right)\begin{vmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & \frac{54}{5} \end{vmatrix} = \frac{5}{3} \cdot \frac{54}{5} = \boxed{18}$$

Evaluation of triangular matrix determinant is trivial!