Lecture #5 Matrices

I. Matrices and Matrix Algebra

- Matrices are valuable in physics for the study of linear equations, linear transformations, quantum mechanics, classical and relativistic mechanics, and particle physics.
- Also valuable for efficient numerical solvers, e.g. Matlab.

A. Basics

1. Linear Algebra

$$a_1x_1 + a_2x_2 = h_1 \qquad b_1x_1 + b_2x_2 = h_2 \Rightarrow \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Two Separate Equations.

2. 2-D, $m \times n$ matrix:

a. (m, n)
$(r, c) \Rightarrow$ RC Order

b. $\underset{\approx}{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

c. $(\underset{\approx}{A})_{ij} = \underset{\text{element}}{\underset{\uparrow}{a_{ij}}}$

d. Square Matrix $(m = n)$

e. Row Vector $\underset{\sim}{a_2} = (a_{21}\, a_{22}\, a_{23})$ Column Vector $\underset{\sim}{a_3} = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$

3. Equality (element by element) $\underset{\approx}{A} = \underset{\approx}{B}$ if $a_{ij} = b_{ij}$ for all i, j.

4. Determinant of Square matrix $\underset{\approx}{A}$ $\quad \det(\underset{\approx}{A})$

5. Addition & Subtraction are applied element by element.

6. Multiplication by a Scalar: a. $\underset{\approx}{B} = \alpha \underset{\approx}{A} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$
b. Multiplies each element: $b_{ij} = \alpha a_{ij}$
c. NOTE: This is different from $\alpha \det(\underset{\approx}{A})$, which only multiplies ~~[illegible]~~ one row or column.

Lecture #5 (Continued)

I. A. (Continued)

7. Matrix Multiplication (Inner Product)

a. $\boxed{\underset{\approx}{A}\,\underset{\approx}{B} = \underset{\approx}{C} \quad \text{where } C_{ij} = \sum_k a_{ik} b_{kj}}$

b. $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{pmatrix} = \begin{pmatrix} \overset{C_{11}}{a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41}} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$(2,4) \longleftrightarrow (4,3)$ — $(2,3)$

To multiply, "inner" indices must match!

Each row multiplies each column

c. NOT commutative (in general) $\underset{\approx}{A}\,\underset{\approx}{B} \neq \underset{\approx}{B}\,\underset{\approx}{A}$

d. For square matrices, Def Commutator: $\boxed{[\underset{\approx}{A}, \underset{\approx}{B}] = \underset{\approx}{A}\,\underset{\approx}{B} - \underset{\approx}{B}\,\underset{\approx}{A}}$

8. Ex: Pauli Matrices (Quantum Mechanics)

$$\underset{\approx}{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \underset{\approx}{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \underset{\approx}{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a. $$\underset{\approx}{\sigma_1}\,\underset{\approx}{\sigma_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

b. $$\underset{\approx}{\sigma_2}\,\underset{\approx}{\sigma_1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix}$$

c. Commutator: $$[\underset{\approx}{\sigma_1}, \underset{\approx}{\sigma_2}] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\,\underset{\approx}{\sigma_3}$$

9. Ex: Different Size Matrices: $\underset{\approx}{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $(3,1)$ $\quad \underset{\approx}{B} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$ $(1,3)$

a. $$\underset{\approx}{A}\,\underset{\approx}{B} = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix}$$

$(3,1)(1,3) \Rightarrow (3,3)$

$$\underset{\approx}{B}\,\underset{\approx}{A} = (4\cdot1 + 5\cdot2 + 6\cdot3) = (32)$$

$(1,3)(3,1) \Rightarrow (1,1)$

Lecture #5 (Continued)

I. (Continued)

B. More Matrix Properties

1a. Def: Unit Matrix $\underset{\approx}{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\approx}{1}_3$ "Identity Matrix"

b. $\underset{\approx}{1}\,\underset{\approx}{A} = \underset{\approx}{A} = \underset{\approx}{A}\,\underset{\approx}{1}$ (Square $\underset{\approx}{A}$)

c. Non-square $\underset{\approx}{A}$ If $\underset{(m,n)}{\underset{\approx}{A}}\;\underset{(n,n)}{\underset{\approx}{1}} = \underset{(m,n)}{\underset{\approx}{A}}$ $\Rightarrow$

2. Def: Diagonal Matrix: $d_{ij} \neq 0$ only for $i=j$, $\underset{\approx}{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

3. Def: Matrix Inverse:

a. If $\underset{\approx}{A}\,\underset{\approx}{B} = \underset{\approx}{1}$, then $\underset{\approx}{B} = \underset{\approx}{A}^{-1}$ is inverse of $\underset{\approx}{A}$.

b. If $\underset{\approx}{A}^{-1}$ exists, it is unique

c. Square matrices only.

d. Not all

d. Def: Singular Matrix Not all non-zero matrices $\underset{\approx}{A}$ have an inverse.

C. Matrix Inversion: Gauss-Jordan Matrix Inversion

1. Compute $\underset{\approx}{M}$ such that $\underset{\approx}{M}\,\underset{\approx}{A} = \underset{\approx}{1}$ $\Rightarrow$ $\underset{\approx}{M} = \underset{\approx}{A}^{-1}$

2. Procedure: $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(1) Divide each row to obtain unity in first column $\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 4 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 \end{pmatrix}$ Do the Same operations to unit Matrix

I. Lecture #5 (Continued)

I. C2. (Continued)

2 Subtract first row from 2nd & 3rd

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{11}{3} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

3 Divide to get 2nd col of 2nd row to unity

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{5} \\ 0 & \frac{1}{3} & \frac{11}{3} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

4 a. Subtract $\frac{2}{3}$ row 2 from row 1

b. Subtract $\frac{1}{3}$ row 2 from row 3

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & \frac{18}{5} \end{pmatrix} \qquad \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ -\frac{1}{5} & -\frac{1}{5} & 1 \end{pmatrix}$$

5 Divide to get 3rd col of 3rd equal to unity

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{18} \end{pmatrix}$$

6 a. Subtract $\frac{1}{5}$ row 3 from rows 1 & 2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boxed{\begin{pmatrix} \frac{11}{18} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{18} \end{pmatrix} = \underset{\approx}{A}^{-1}}$$

$$\underset{\approx}{M}\,\underset{\approx}{A} = \underset{\approx}{1} \qquad \underset{\approx}{M}\,\underset{\approx}{1} = \underset{\approx}{M}$$

D. More Matrix Properties

1. Derivatives of Determinants: $\boxed{\frac{d}{dt}\left[\det(\underset{\approx}{A})\right] = \det(\underset{\approx}{A}) \sum_{ij} \left(\underset{\approx}{A}^{-1}\right)_{ji} \frac{da_{ij}}{dt}}$

2. Solving Systems of Linear Equations

a. $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{h}$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{n \times n \text{ matrix} \Rightarrow n \text{ equations, } n \text{ unknowns!}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

I. D. 2. (Continued)

b. Multiply on left by $\underset{\approx}{A}^{-1}$ (only if $\underset{\approx}{A}$ is non-singular)

$$\underbrace{[\underset{\approx}{A}^{-1}\underset{\approx}{A}]}_{=\underset{\approx}{1}}\underset{\sim}{x} = \underset{\approx}{A}^{-1}\underset{\sim}{b} \Rightarrow \boxed{\underset{\sim}{x} = \underset{\approx}{A}^{-1}\underset{\sim}{b}}$$

c. NOTE: i. If we can evaluate $\underset{\approx}{A}^{-1}$, we can compute $\underset{\sim}{x}$ solution.

ii. Existence of $\underset{\approx}{A}^{-1}$ means $\underset{\sim}{x}$ is a unique solution.
Also when $\det(\underset{\approx}{A}) \neq 0$

d. $\boxed{\text{A square matrix } \underset{\approx}{A} \text{ is singular if and only if } \det(\underset{\approx}{A}) = 0}$

3. Determinant Product Theorem $\boxed{\det(\underset{\approx}{A}\underset{\approx}{B}) = \det(\underset{\approx}{A})\det(\underset{\approx}{B})}$

4.a. Def: Transpose $\underset{\approx}{A}^T$ $\quad (\underset{\approx}{A}^T)_{ij} = a_{ji}$ (also $\tilde{\underset{\approx}{A}}$)
(Swap rows & columns)

b. $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \underset{\sim}{x}^T = (x_1 \ x_2 \ x_3)$

c. Def: Symmetric Matrix if $\underset{\approx}{A}^T = \underset{\approx}{A}$

5. Complex Conjugate $\underset{\approx}{A}^*$ $\quad$ If $(\underset{\approx}{A})_{ij} = a_{ij}$, then $(\underset{\approx}{A}^*)_{ij} = a_{ij}^*$
(element by element)

6. Def: Adjoint $\underset{\approx}{A}^\dagger$ $\quad (\underset{\approx}{A}^\dagger)_{ij} = a_{ji}^*$

(Both complex conjugation and transposition, in either order)

Lecture #5 (Continued)

I. D. (Continued)

7a. Def: Trace $\quad \text{Trace}(\underset{\approx}{A}) = \sum_{i=1}^{n} a_{ii}$

(Sum of diagonal elements)

b. $\text{Trace}(\underset{\approx}{A} + \underset{\approx}{B}) = \text{trace}(\underset{\approx}{A}) + \text{trace}(\underset{\approx}{B})$

c. $\text{trace}(\underset{\approx}{A}\underset{\approx}{B}) = \text{trace}(\underset{\approx}{B}\underset{\approx}{A})$, even if $\underset{\approx}{A}\underset{\approx}{B} \neq \underset{\approx}{B}\underset{\approx}{A}$

8. Matrix Multiplication and other operations

a. Remember, i. $\det(\underset{\approx}{A}\underset{\approx}{B}) = \det(\underset{\approx}{A})\det(\underset{\approx}{B}) = \det(\underset{\approx}{B}\underset{\approx}{A})$

ii. $\text{trace}(\underset{\approx}{A}\underset{\approx}{B}) = \text{trace}(\underset{\approx}{B}\underset{\approx}{A})$

b. Transpose: $(\underset{\approx}{A}\underset{\approx}{B})^T = \underset{\approx}{B}^T\underset{\approx}{A}^T$

c. Adjoint: $(\underset{\approx}{A}\underset{\approx}{B})^\dagger = \underset{\approx}{B}^\dagger\underset{\approx}{A}^\dagger$

d. Inverse: $(\underset{\approx}{A}\underset{\approx}{B})^{-1} = \underset{\approx}{B}^{-1}\underset{\approx}{A}^{-1}$

9. Connection to Vectors: a. $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

b. Dot Product for vectors: $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\Rightarrow \underset{\sim}{a}^T \underset{\sim}{b} = (a_1\ a_2\ a_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

c. Magnitude Squared: $\underset{\sim}{a}^\dagger \underset{\sim}{a} \Longleftrightarrow |\underset{\sim}{a}|^2$

10a. Def: Orthogonal Matrices $\quad \underset{\approx}{S}^{-1} = \underset{\approx}{S}^T$ (Transpose equals inverse)

b. $\Rightarrow \underset{\approx}{S}\underset{\approx}{S}^T = \underset{\approx}{1}$

c. $\det(\underset{\approx}{S}) = \pm 1$

Lecture #5 (Continued)

I. D. (Continued)

11\. a. Def: Unitary Matrix $\underset{\approx}{U}^{\dagger} = \underset{\approx}{U}^{-1}$ Adjoint equals inverse

b. $\underset{\approx}{U}\underset{\approx}{U}^{\dagger} = \underset{\approx}{U}^{\dagger}\underset{\approx}{U} = \underset{\approx}{1}$

c. We may write $\det(\underset{\approx}{U}) = e^{i\theta}$, $\det(\underset{\approx}{U}^{\dagger}) = e^{-i\theta}$

d. If $\underset{\approx}{U}$ & $\underset{\approx}{V}$ are unitary, $\underset{\approx}{U}\underset{\approx}{V}$ is also unitary.

12\. a. Def: Hermitian Matrix (Self-Adjoint) $\underset{\approx}{H} = \underset{\approx}{H}^{\dagger}$

b. Thus $a_{ji}^{*} = a_{ij}$ (Reflected about diagonal → complex conjugates)

c. So diagonal elements $a_{ii} = a_{ii}^{*}$ must be real.

d. $\boxed{\text{All real, symmetric matrices are self-adjoint.}}$

e. Def: Anti-Hermitian: If $\underset{\approx}{A}\underset{\approx}{B} - \underset{\approx}{B}\underset{\approx}{A} \neq 0$,

$$(\underset{\approx}{A}\underset{\approx}{B} - \underset{\approx}{B}\underset{\approx}{A})^{\dagger} = \underset{\text{"anti"}}{-}(\underset{\approx}{A}\underset{\approx}{B} - \underset{\approx}{B}\underset{\approx}{A})$$

13\. Unit Vectors:

a. $\hat{\underset{\sim}{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{\underset{\sim}{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

b. To extract a row or column:

$$\underset{\approx}{A}\hat{\underset{\sim}{e}}_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$$

$$\hat{\underset{\sim}{e}}_2^{T}\underset{\approx}{A} = (a_{21} \;\; a_{22} \;\; a_{23})$$

I. D. (Continued)

14.a. Direct Product $\underset{\approx}{C} = \underset{\approx}{A} \otimes \underset{\approx}{B}$ where $C_{\alpha\beta} = A_{ij} B_{kl}$, with $\underset{\approx}{A}$ an (m,n) matrix and $\underset{\approx}{B}$ an (m',n') matrix

$$\alpha = m'(i-1)+k \quad , \quad \beta = n'(j-1)+l$$

b. $\underset{\approx}{C}$ is an (mm', nn') matrix

c. Ex: $\underset{\approx}{A}$ & $\underset{\approx}{B}$ are 2×2 matrices

$$\underset{\approx}{A} \otimes \underset{\approx}{B} = \begin{pmatrix} a_{11}\underset{\approx}{B} & a_{12}\underset{\approx}{B} \\ a_{21}\underset{\approx}{B} & a_{22}\underset{\approx}{B} \end{pmatrix} = \left(\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{22}b_{22} \\ \hline a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{array}\right)$$

E. Functions of Matrices (element by element)

1. $$\exp(\underset{\approx}{A}) = \sum_{j=0}^{\infty} \frac{1}{j!} (\underset{\approx}{A})^j$$

2. Euler Identity for Pauli Matrices,

$$e^{i\sigma_k \theta} = \underset{\approx}{1}_2 \cos\theta + i\sigma_k \sin\theta$$

3. Hermitian & Unitary Matrices: $\underset{\approx}{U} = e^{i\underset{\approx}{H}}$

a. Take adjoint: $\underset{\approx}{U}^{\dagger} = e^{-i\underset{\approx}{H}^{\dagger}} = e^{-i\underset{\approx}{H}} = \left[e^{i\underset{\approx}{H}}\right]^{-1} = \underset{\approx}{U}^{-1}$

4. Trace Formula: $\det[\exp(\underset{\approx}{H})] = \exp[\mathrm{trace}(\underset{\approx}{H})]$