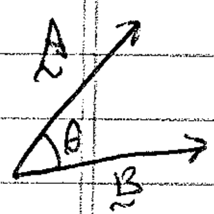
Hawes ①

# Lecture #6 Vector Analysis: Basics and Transformations

## I. Vector Analysis

### A. Basic Properties

1. A vector $\underset{\sim}{A}$ has magnitude and direction
2. Components: In terms of unit vectors specifying a coordinate system
$$\underset{\sim}{A} = A_x \hat{\underset{\sim}{e}}_x + A_y \hat{\underset{\sim}{e}}_y + A_z \hat{\underset{\sim}{e}}_z$$
3. Magnitude
$$|\underset{\sim}{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
4. Dot Product




a. $\underset{\sim}{A} \cdot \underset{\sim}{B} = A_x B_x + A_y B_y + A_z B_z$

b. $\underset{\sim}{A} \cdot \underset{\sim}{B} = |\underset{\sim}{A}||\underset{\sim}{B}| \cos\theta$

c. If $\underset{\sim}{A} \cdot \underset{\sim}{B} = 0$, vectors are orthogonal

d. Projection: $\hat{\underset{\sim}{e}}_x \cdot \underset{\sim}{A} = A_x(\hat{\underset{\sim}{e}}_x \cdot \hat{\underset{\sim}{e}}_x) + A_y \cancelto{0}{(\hat{\underset{\sim}{e}}_x \cdot \hat{\underset{\sim}{e}}_y)} + A_z \cancelto{0}{(\hat{\underset{\sim}{e}}_x \cdot \hat{\underset{\sim}{e}}_z)}$
$$= A_x$$

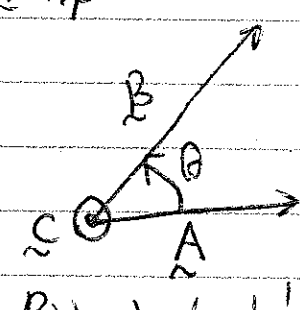
5. Column Vectors (Matrix Notation) a. $\underset{\sim}{A} \Rightarrow \underset{\sim}{a} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

b. $\underset{\sim}{a}^T \underset{\sim}{b} \Leftrightarrow \underset{\sim}{A} \cdot \underset{\sim}{B}$

### B. Cross Product

1. Useful for angular momentum $\underset{\sim}{L} = \underset{\sim}{r} \times \underset{\sim}{p}$
2. Def: Cross Product

a. $\underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B} = A B \sin\theta \, \hat{\underset{\sim}{e}}_c$



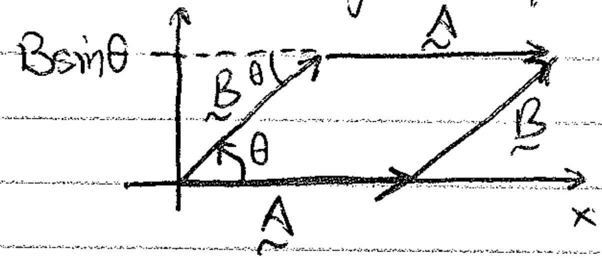

b. $\hat{\underset{\sim}{e}}_c$ is perpendicular to the plane defined by $\underset{\sim}{A}$ & $\underset{\sim}{B}$.

Right-hand rule!

"Anthropomorphic prescription"

I. B. Continued

3. a. $\underset{\sim}{A} \times \underset{\sim}{B}$ has magnitude equal to area of parallelogram




b. Direction of $\underset{\sim}{A} \times \underset{\sim}{B}$ is normal to parallelogram.

4. <u>Anti-Commutation:</u> $\boxed{\underset{\sim}{A} \times \underset{\sim}{B} = -\underset{\sim}{B} \times \underset{\sim}{A}}$

5. <u>Distributive:</u> a. $\underset{\sim}{A} \times (\underset{\sim}{B} + \underset{\sim}{C}) = \underset{\sim}{A} \times \underset{\sim}{B} + \underset{\sim}{A} \times \underset{\sim}{C}$

b. $k(\underset{\sim}{A} \times \underset{\sim}{B}) = (k\underset{\sim}{A}) \times \underset{\sim}{B}$

6. <u>Unit Vectors:</u> a. $\boxed{\hat{\underset{\sim}{e}}_i \times \hat{\underset{\sim}{e}}_j = \sum_k \varepsilon_{ijk} \hat{\underset{\sim}{e}}_k}$

b. Thus
$$\hat{\underset{\sim}{e}}_x \times \hat{\underset{\sim}{e}}_y = \hat{\underset{\sim}{e}}_z \qquad \hat{\underset{\sim}{e}}_y \times \hat{\underset{\sim}{e}}_x = -\hat{\underset{\sim}{e}}_z$$
$$\hat{\underset{\sim}{e}}_y \times \hat{\underset{\sim}{e}}_z = \hat{\underset{\sim}{e}}_x \qquad \text{etc.}$$
$$\hat{\underset{\sim}{e}}_z \times \hat{\underset{\sim}{e}}_x = \hat{\underset{\sim}{e}}_y$$

7. <u>Component Form:</u>

a. $$\underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B} = \underbrace{(A_x B_y - A_y B_x)}_{=C_z}\hat{\underset{\sim}{e}}_z + \underbrace{(A_z B_x - A_x B_z)}_{=C_y}\hat{\underset{\sim}{e}}_y + \underbrace{(A_y B_z - A_z B_y)}_{=C_x}\hat{\underset{\sim}{e}}_x$$

b. In general, $\boxed{C_i = \sum_{jk} \varepsilon_{ijk} A_j B_k}$

c. <u>Determinant Form:</u>
$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{vmatrix} \hat{\underset{\sim}{e}}_x & \hat{\underset{\sim}{e}}_y & \hat{\underset{\sim}{e}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Can expand in minors about top row.

8. The Cross Product is rotationally invariant (it does not depend on the coordinate system chosen).

9. The Cross Product is a specifically 3-D quantity!

## C. Scalar Triple Product

1. $\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

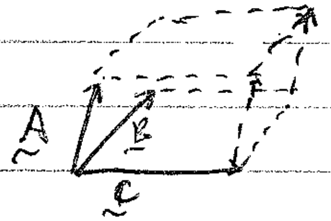
   a. Yields a scalar quantity!
   b. Must be rotationally invariant (since dot and cross products are rotationally invariant)

2. Sign changes for odd permutations; no change for even permutations.

$$\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C} = \underset{\sim}{B} \cdot \underset{\sim}{C} \times \underset{\sim}{A} = \underset{\sim}{C} \cdot \underset{\sim}{A} \times \underset{\sim}{B} = -\underset{\sim}{A} \cdot \underset{\sim}{C} \times \underset{\sim}{B} \text{ etc.}$$

NOTE: Can drop parentheses around cross product
⇒ always evaluate cross product first!

3. Volume of parallelopiped defined by $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$.




$$\text{Volume} = |\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C}|$$

D. Vector Triple Product $\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C})$

1. NOTE: Parentheses are essential here since it is often true that $(\underset{\sim}{A} \times \underset{\sim}{B}) \times \underset{\sim}{C} \neq \underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C})$!

2. $$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = \underset{\sim}{B}(\underset{\sim}{A} \cdot \underset{\sim}{C}) - \underset{\sim}{C}(\underset{\sim}{A} \cdot \underset{\sim}{B})$$

"BAC - CAB" Rule!

3. Proof using index notation:

a. Remember definition $\boxed{C_i = \sum_{jk} \varepsilon_{ijk} A_j B_k}$ with $\boxed{\underset{\sim}{C} = \sum_i \hat{\underset{\sim}{e}}_i C_i}$

and use relation $\boxed{\sum_k \varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}}$

b. Thus

$$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = \sum_i \hat{\underset{\sim}{e}}_i \sum_{jk} \varepsilon_{ijk} A_j \left[\sum_{pq} \varepsilon_{kpq} B_p C_q\right]$$

$$= \sum_{ij} \sum_{pq} \hat{\underset{\sim}{e}}_i A_j B_p C_q \left(\sum_k \varepsilon_{ijk} \varepsilon_{kpq}\right)$$

c. NOTE: $\varepsilon_{kpq} = -\varepsilon_{pkq} = +\varepsilon_{pqk}$

d. Thus

$$= \sum_{ij} \sum_{pq} \hat{\underset{\sim}{e}}_i A_j B_p C_q \left[\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right]$$

e. Eliminate p,q sums:

$$= \sum_{ij} \hat{e}_i A_j (B_i C_j - B_j C_i) = \sum_i \hat{\underset{\sim}{e}}_i \left[B_i \left(\sum_j A_j C_j\right) - C_i \left(\sum_j A_j B_j\right)\right]$$

f. Using $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_j A_j B_j$, we obtain $= \sum_i \hat{\underset{\sim}{e}}_i \left[B_i (\underset{\sim}{A} \cdot \underset{\sim}{C}) - C_i (\underset{\sim}{A} \cdot \underset{\sim}{B})\right]$

I. (Continued)

D. 3. (Continued)

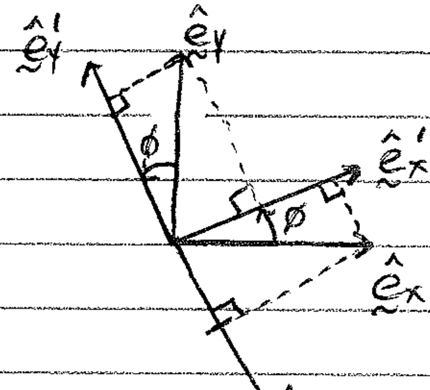
g. Finally $= \underset{\sim}{B}(\underset{\sim}{A}\cdot\underset{\sim}{C}) - \underset{\sim}{C}(\underset{\sim}{A}\cdot\underset{\sim}{B})$ ✓

## II. Coordinate Transformations

- Vectors must have specific transformation properties under rotation of the coordinate system.

### A. Rotations in 2D ($\mathbb{R}^2$)

1. Consider rotation to a new (primed) coordinate system




a. $\underset{\sim}{A} = A_x \hat{\underset{\sim}{e}}_x + A_y \hat{\underset{\sim}{e}}_y$

$= A_x' \hat{\underset{\sim}{e}}_x' + A_y' \hat{\underset{\sim}{e}}_y'$

b. How are $A_x'$ & $A_y'$ related to $A_x$ & $A_y$?

2. First, represent $\hat{\underset{\sim}{e}}_x$ and $\hat{\underset{\sim}{e}}_y$ in new coordinate system.

a. $\hat{\underset{\sim}{e}}_x = \cos\phi\, \hat{\underset{\sim}{e}}_x' - \sin\phi\, \hat{\underset{\sim}{e}}_y'$

b. $\hat{\underset{\sim}{e}}_y = \sin\phi\, \hat{\underset{\sim}{e}}_x' + \cos\phi\, \hat{\underset{\sim}{e}}_y'$

3. Simply substitute: $\underset{\sim}{A} = A_x(\cos\phi\, \hat{\underset{\sim}{e}}_x' - \sin\phi\, \hat{\underset{\sim}{e}}_y') + A_y(\sin\phi\, \hat{\underset{\sim}{e}}_x' + \cos\phi\, \hat{\underset{\sim}{e}}_y')$

$$= \underbrace{(A_x\cos\phi + A_y\sin\phi)}_{=A_x'} \hat{\underset{\sim}{e}}_x' + \underbrace{(-A_x\sin\phi + A_y\cos\phi)}_{=A_y'} \hat{\underset{\sim}{e}}_y'$$

4. Matrix Notation: $\cos\phi\, A_x + \sin\phi\, A_y = A_x'$

$-\sin\phi\, A_x + \cos\phi\, A_y = A_y'$

$$\boxed{\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} A_x' \\ A_y' \end{pmatrix}}$$

II. A. (Continued)

5. For reverse transformation $\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x' \\ A_y' \end{pmatrix}$

6. Define Matrices $\underset{\approx}{S} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ $\quad \underset{\approx}{S}' = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

where $\underset{\sim}{A}' = \underset{\approx}{S}\,\underset{\sim}{A}$ and $\underset{\sim}{A} = \underset{\approx}{S}'\underset{\sim}{A}'$.

7. a. Thus $\underset{\sim}{A}' = \underset{\approx}{S}\left[\underset{\approx}{S}'\underset{\sim}{A}'\right] \Rightarrow \underset{\sim}{A}' = \left(\underset{\approx}{S}\,\underset{\approx}{S}'\right)\underset{\sim}{A}'$

b. This must be valid for any $\underset{\sim}{A}'$, so $\boxed{\underset{\approx}{S}' = \underset{\approx}{S}^{-1}}$

Such that $\underset{\approx}{S}\,\underset{\approx}{S}' = \underset{\approx}{1}$.

c. Also, by inspection $\boxed{\underset{\approx}{S}' = \underset{\approx}{S}^T}$

d. Thus, since $\boxed{\underset{\approx}{S}^{-1} = \underset{\approx}{S}^T}$, $\underset{\approx}{S}$ is an <u>orthogonal matrix!</u>

B. <u>Orthogonal Transformations</u>

1. We can write $\underset{\sim}{\hat{e}_x} = \underbrace{(\underset{\sim}{\hat{e}_x'}\cdot\underset{\sim}{\hat{e}_x})}_{\text{Projection onto } \hat{e}_x'}\underset{\sim}{\hat{e}_x'} + \underbrace{(\underset{\sim}{\hat{e}_y'}\cdot\underset{\sim}{\hat{e}_x})}_{\text{Projection onto } \hat{e}_y'}\underset{\sim}{\hat{e}_y'}$

2. Thus $\underset{\approx}{S} = \begin{pmatrix} \hat{e}_x'\cdot\hat{e}_x & \hat{e}_x'\cdot\hat{e}_y \\ \hat{e}_y'\cdot\hat{e}_x & \hat{e}_y'\cdot\hat{e}_y \end{pmatrix}$

3. Deeper meaning of an orthogonal matrix!

a. Each row contains components of unit vectors $\underset{\sim}{\hat{e}_x'}$ & $\underset{\sim}{\hat{e}_y'}$ on the unprimed coordinate axes.

b. Dot product of different rows $= 0$

c. Dot product of any row with itself $= 1$

d. Similarly, each column contains components of $\underset{\sim}{\hat{e}_x}$ & $\underset{\sim}{\hat{e}_y}$ on the primed coordinate axes. Same result for dot products of columns!

e. Consistent with $\underset{\approx}{S}\,\underset{\approx}{S}^T = \underset{\approx}{1}$

4. $\boxed{\text{Transformation from one Cartesian coordinate system to another is } \underline{\text{orthogonal}}!}$

II. (Continued)



C. Reflections

1. Def: Inversion a. $\underset{\approx}{S} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, b. $\det(\underset{\approx}{S}) = -1$.

c. Conversion from right-handed to left-handed system.

2. Def: Reflection about a plane a. $\underset{\approx}{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ about x-y plane.

b. $\det(\underset{\approx}{S}) = -1$ c. Also RH to LH coordinates

3. Vectors vs. Pseudovectors:

a. Vector addition, multiplication by scalar, & dot product unaffected by reflection

b. But, for reflection or inversion, cross product changes sign.

c. Polar vectors $\underset{\sim}{B}$ $\underset{\sim}{C}$ $\Rightarrow \underset{\sim}{B}' = \underset{\approx}{S}\underset{\sim}{B}$ $\Rightarrow$ Vector

Axial vectors $\underset{\sim}{B} \times \underset{\sim}{C}$ $\Rightarrow (\underset{\sim}{B}\times\underset{\sim}{C}) = \det(\underset{\approx}{S})\,\underset{\approx}{S}\,(\underset{\sim}{B}\times\underset{\sim}{C}) \Rightarrow$ pseudovector

D. Order of Operations

1. Successive rotations: $\underset{\sim}{A}' = \underset{\approx}{S}(R')\,\underset{\approx}{S}(R)\,\underset{\sim}{A}$

a. Operations from right to left.

b. Product matrix $\underset{\approx}{S}(R'R) = \underset{\approx}{S}(R')\underset{\approx}{S}(R)$ is orthogonal.

E. Rotations in 3D ($\mathbb{R}^3$)

1. $$\underset{\approx}{S} = \begin{pmatrix} \hat{\underset{\sim}{e}}_1'\cdot\hat{\underset{\sim}{e}}_1 & \hat{\underset{\sim}{e}}_1'\cdot\hat{\underset{\sim}{e}}_2 & \hat{\underset{\sim}{e}}_1'\cdot\hat{\underset{\sim}{e}}_3 \\ \hat{\underset{\sim}{e}}_2'\cdot\hat{\underset{\sim}{e}}_1 & \hat{\underset{\sim}{e}}_2'\cdot\hat{\underset{\sim}{e}}_2 & \hat{\underset{\sim}{e}}_2'\cdot\hat{\underset{\sim}{e}}_3 \\ \hat{\underset{\sim}{e}}_3'\cdot\hat{\underset{\sim}{e}}_1 & \hat{\underset{\sim}{e}}_3'\cdot\hat{\underset{\sim}{e}}_2 & \hat{\underset{\sim}{e}}_3'\cdot\hat{\underset{\sim}{e}}_3 \end{pmatrix}$$ element $S_{\mu\nu} = \hat{\underset{\sim}{e}}_\mu'\cdot\hat{\underset{\sim}{e}}_\nu$

2. Projection of $\hat{\underset{\sim}{e}}_\mu$ onto $\hat{\underset{\sim}{e}}_\nu'$:

a. Describe change in $x_\nu$ produced by unit change in $x_\mu'$ $\Rightarrow \boxed{\frac{\partial x_\nu}{\partial x_\mu'}}$

b. Thus, $\underset{\approx}{S}$ can be written in terms of $\frac{\partial x_\nu}{\partial x_\mu'}$ For linear relation only!

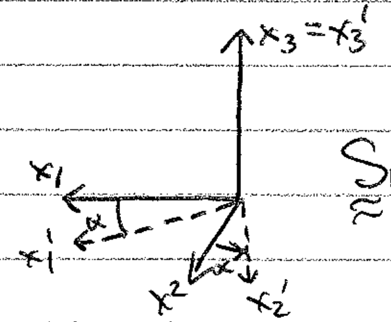
c. Linear restriction means only true among Cartesian coordinate systems.

3\. a. In $\mathbb{R}^3$, we require 3 angles to specify arbitrary rotation ( 1 angle in $\mathbb{R}^2$)

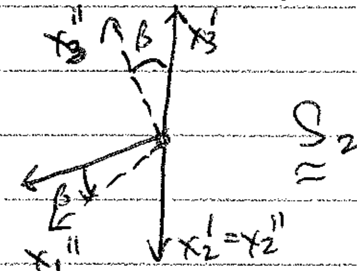
b. Thus, of 9 elements in $\underset{\approx}{S}$, only <u>three</u> are independent.

4\. <u>Specify 3 successive rotations:</u>

a. $\underset{\approx}{S}_1(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\boxed{\text{Rotation about } x_3}$

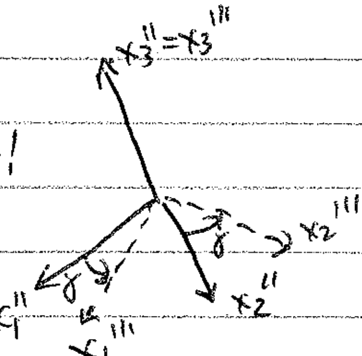



b. $\underset{\approx}{S}_2(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$ $\boxed{\text{Rotation about } x_2'}$




c. $\underset{\approx}{S}_3(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\boxed{\text{Rotation about } x_3''}$

d. Total $\boxed{\underset{\approx}{S}(\alpha,\beta,\gamma) = \underset{\approx}{S}_3(\gamma)\underset{\approx}{S}_2(\beta)\underset{\approx}{S}_1(\alpha)}$ Order is <u>important!</u>




5\. Total $\mathbb{R}^3$ Rotation

$$\boxed{\underset{\approx}{S}(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix}}$$

a. Elements are $\hat{\underset{\sim}{e}}_i''' \cdot \hat{\underset{\sim}{e}}_j$

b. $\det(\underset{\approx}{S}) = +1$